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Structural stability of a dynamical system near a non-hyperbolic fixed point

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Abstract

We prove structural stability under perturbations for a class of discrete-time dynamical systems near a non-hyperbolic fixed point. We reformulate the stability problem in terms of the well-posedness of an infinite-dimensional nonlinear ordinary differential equation in a Banach space of carefully weighted sequences. Using this, we prove existence and regularity of flows of the dynamical system which obey mixed initial and final boundary conditions. The class of dynamical systems we study, and the boundary conditions we impose, arose in a renormalisation group analysis of the 4-dimensional weakly self-avoiding walk.

1 Introduction and main result

1.1 Introduction

Let $\mathcal{V} = \mathbb{R}^3$ with elements $V \in \mathcal{V}$ written $V = (g, z, \mu)$. For each $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we define the *quadratic flow* $\bar{\varphi}_j : \mathcal{V} \to \mathcal{V}$ by

$$\bar{\varphi}_j(V) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta_j & \gamma_j & \lambda_j \end{pmatrix} V - \begin{pmatrix} V^T q_j^g V \\ V^T q_j^z V \\ V^T q_j^\mu V \end{pmatrix}, \tag{1.1}$$

with the quadratic terms of the form

$$q_j^g = \begin{pmatrix} \beta_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_j^z = \begin{pmatrix} \theta_j & \frac{1}{2}\zeta_j & 0 \\ \frac{1}{2}\zeta_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.2}$$

and

$$q_j^\mu = \begin{pmatrix} \upsilon_j^{gg} & \frac{1}{2}\upsilon_j^{gz} & \frac{1}{2}\upsilon_j^{g\mu} \\ \frac{1}{2}\upsilon_j^{gz} & \upsilon_j^{zz} & \frac{1}{2}\upsilon_j^{z\mu} \\ \frac{1}{2}\upsilon_j^{g\mu} & \frac{1}{2}\upsilon_j^{z\mu} & 0 \end{pmatrix}. \tag{1.3}$$

All entries in the above matrices are real numbers. We assume that there exists λ such that $\lambda_j \geq \lambda > 1$ for all j, together with assumptions that ensure that for most values of j we have $\beta_j \geq c > 0$ and $\zeta_j \leq 0$. Our hypotheses on the parameters of $\bar{\varphi}$ are stated precisely in Assumptions (A1–A2) below.

The quadratic flow $\bar{\varphi}$ defines a time-dependent discrete-time 3-dimensional dynamical system. It is triangular, in the sense that the equation for g does not depend on z or μ, the equation for z depends only on g, and the equation for μ depends on g and z. Moreover, the equation for z is linear in z, and the equation for μ is linear in μ. This makes the analysis of the quadratic flow elementary.

Our main result concerns structural stability of $\bar{\varphi}$ under a class of infinite-dimensional perturbations. Let $(\mathcal{W}_j)_{j\in\mathbb{N}_0}$ be a sequence of Banach spaces and $X_j = \mathcal{W}_j \oplus \mathcal{V}$. We write $x_j \in X_j$ as $x_j = (K_j, V_j) = (K_j, g_j, z_j, \mu_j)$. The norm on X_j is given by

$$\|x_j\|_{X_j} = \max\{\|K_j\|_{\mathcal{W}_j}, \|V\|_{\mathcal{V}}\} = \max\{\|K_j\|_{\mathcal{W}_j}, |g|, |z|, |\mu|\}. \tag{1.4}$$

Suppose that we are given maps $\psi_j : X_j \to \mathcal{W}_{j+1}$ and $\rho_j : X_j \to \mathcal{V}$. Then we define $\Phi_j : X_j \to X_{j+1}$ by

$$\Phi_j(K_j, V_j) = (\psi_j(K_j, V_j), \bar{\varphi}_j(V_j) + \rho_j(K_j, V_j)). \tag{1.5}$$

This is an infinite-dimensional perturbation of the 3-dimensional quadratic flow $\bar{\varphi}$, which breaks triangularity and which involves the spaces $\mathcal{W}_j$ in a nontrivial way. We will impose estimates on ψ_j and ρ_j below, which make Φ a third-order perturbation of $\bar{\varphi}$.

We give hypotheses under which there exists a sequence $(x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$ which is a *global flow* of Φ, in the sense that

$$x_{j+1} = \Phi_j(x_j) \quad \text{for all } j \in \mathbb{N}_0, \tag{1.6}$$

obeying the boundary conditions that (K_0, g_0) is fixed, $z_j \to 0$, and $\mu_j \to 0$. Moreover, within an appropriate space of sequences, this global flow is unique.

As we discuss in more detail in Section 1.4 below, this result provides an essential ingredient in a renormalisation group analysis of the 4-dimensional continuous-time weakly self-avoiding walk [2, 4], where the boundary condition $\lim_{j\to\infty} \mu_j = 0$ is the appropriate boundary condition for the study of a *critical* trajectory. It is this application that provides our immediate motivation to study the dynamical system Φ, but we expect that the methods developed here will have further applications to dynamical systems arising in renormalisation group analyses.

1.2 Dynamical system

We think of $\Phi = (\Phi_j)_{j\in\mathbb{N}_0}$ as the *evolution map* of a discrete time-dependent dynamical system, although it is more usual in dynamical systems to have the spaces X_j be identical. Our application in [2, 4] requires the greater generality of j-dependent spaces.

In the case that Φ is a time-independent dynamical system, i.e., when $\Phi_j = \Phi$ and $X_j = X$ for all $j \in \mathbb{N}_0$, its fixed points are of special interest: $x^* \in X$ is a fixed point of Φ if $x^* = \Phi(x^*)$. The dynamical system is called *hyperbolic* near a fixed point $x^* \in X$ if the spectrum of $D\Phi(x^*)$ is disjoint from the unit circle [11]. It is a classic result that for a hyperbolic system there exists a splitting $X \cong X_s \oplus X_u$ into a *stable* and an *unstable manifold* near x^*. The stable manifold is a submanifold $X_s \subset X$ such that $x_j \to x^*$ in X, exponentially fast, when (x_j) satisfies (1.6) and $x_0 \in X_s$. This result can be generalised without much difficulty to the situation when the Φ_j and X_j are not necessarily identical, viewing "0" as a fixed point (although 0 is the origin in different spaces X_j). The hyperbolicity condition must now be imposed in a uniform way [3, Theorem 2.16].

By definition, $\bar{\varphi}_j(0) = 0$, and we will assume below that also $\psi_j(0) = 0$ and $\rho_j(0) = 0$. The dynamical system defined by (1.5) is not hyperbolic near the fixed point 0, due to the two unit eigenvalues of the matrix in the first term of (1.1). Thus the g- and z-directions are *centre* directions, which neither contract nor expand in a linear approximation. On the other hand, the hypothesis that $\lambda_j \geq \lambda > 1$ ensures that the μ-direction is *expanding*, and we will assume below that $\psi_j : X_j \to \mathcal{W}_{j+1}$ is such that the K-direction is *contractive* near the fixed point 0. The behaviour of dynamical systems near non-hyperbolic fixed points is much more subtle than for the hyperbolic case. A general classification does not exist, and a nonlinear analysis is required.

1.3 Main result

In Section 2, we give an elementary proof that there exists a unique global flow $\bar{V} = (\bar{g}, \bar{z}, \bar{\mu})$ of the quadratic flow $\bar{\varphi}$ with boundary conditions $\bar{g}_0 = g_0$, $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$, where we are writing, e.g., $\bar{z}_\infty = \lim_{j\to\infty} \bar{z}_j$. Our main result is that, under the assumptions stated below, there exists a unique global flow of Φ with small initial conditions (K_0, g_0) and final conditions $(z_\infty, \mu_\infty) = (0, 0)$, and that this flow is a small perturbation of $\bar{V}$.

The sequence $\bar{g} = (\bar{g}_j)$ plays a prominent role in the analysis. Determined by the sequence (β_j), it obeys

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \qquad \bar{g}_0 = g_0 > 0. \tag{1.7}$$

We regard $\bar{g}$ as a known sequence (only dependent on the initial condition g_0). The following examples are helpful to keep in mind.

Example 1.1. (i) Constant $\beta_j = b > 0$. In this case, it is not difficult to show that $\bar{g}_j \sim g_0(1 + g_0 bj)^{-1} \sim (bj)^{-1}$ as $j \to \infty$ (e.g., by applying (2.9) below with $\psi(t) = t^{-2}$).
(ii) Abrupt cut-off, with $\beta_j = b$ for $j \leq J$ and $\beta_j = 0$ for $j > J$, with $J \gg 1$. In this case, $\bar{g}_j$ is approximately the constant $(bJ)^{-1}$ for $j > J$. In particular, $\bar{g}_j$ does not go to zero as $j \to \infty$.

Example 1.1 prompts us to make the following general definition of a cut-off time for bounded sequences β_j. Let $\|\beta\|_\infty = \sup_{j\geq 0} |\beta_j| < \infty$, and let $n_+ = n$ if $n \geq 0$ and otherwise $n_+ = 0$. Given a fixed $\Omega > 1$, we define the Ω-*cut-off time* j_Ω by

$$j_\Omega = \inf\{k \geq 0 : |\beta_j| \leq \Omega^{-(j-k)_+} \|\beta\|_\infty \text{ for all } j \geq 0\}. \tag{1.8}$$

The infimum of the empty set is defined to equal ∞, e.g., if $\beta_j = b$ for all j. By definition, $j_\Omega \le j_{\Omega'}$ if $\Omega \le \Omega'$. To abbreviate the notation, we write

$$\chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{1.9}$$

The evolution maps Φ_j are specified by the real parameters η_j, γ_j, λ_j, β_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$, together with the maps ψ_j and ρ_j. We fix $\Omega > 1$ and make the following assumptions (A1–A2) on the real parameters, and (A3) on the maps. Assumption (A4) allows Φ_j to depend continuously on an external parameter. Note that when $j_\Omega < \infty$, (A1) permits the possibility that eventually $\beta_k = 0$ for large k. The simplest setting for the assumptions is for the case $j_\Omega = \infty$, for which $\chi_j = 1$ for all j.

Assumptions.

(A1) *The sequence β:* The sequence (β_j) is bounded: $\|\beta\|_\infty < \infty$. There exists $c > 0$ such that $\beta_j \ge c$ for all but c^{-1} values of $j \le j_\Omega$.

(A2) *The other parameters of $\bar{\varphi}$:* There exists $\lambda > 1$ such that $\lambda_j \ge \lambda$ for all j. There exists $c > 0$ such that $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$. Each of ζ_j, η_j, γ_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$ is bounded in absolute value by $O(\chi_j)$, with a constant that is independent of both j and j_Ω.

(A3) *The perturbation:* The maps $\psi_j : X_j \to \mathcal{W}_{j+1} \subset X_{j+1}$ and $\rho_j : X_j \to \mathcal{V} \subset X_{j+1}$ are three times continuously Fréchet differentiable, and obey $\psi_j(0) = 0$ and $\rho_j(0) = 0$ for all j. There exist $\kappa \in (0, \Omega^{-1})$ such that for all $x_j \in X_j$,

$$\|D_K \psi_j(x_j)\|_{L(\mathcal{W}_j, \mathcal{W}_{j+1})} \le \kappa, \tag{1.10}$$

and there is $M > 0$ such that for both $\phi = \psi$ and $\phi = \rho$,

$$\|D_V \phi_j(x_j)\|_{L(\mathcal{V}, X_{j+1})} \le M \chi_j \|V_j\|_{\mathcal{V}}^2, \tag{1.11}$$

and such that for $m = 2, 3$,

$$\|D_V^m \phi_j(x_j)\|_{L^m(X_j, X_{j+1})} \le M \chi_j, \tag{1.12}$$
$$\|D^m \phi_j(x_j)\|_{L^m(X_j, X_{j+1})} \le M, \tag{1.13}$$

where $D_\alpha \phi$ is the Fréchet derivative of ϕ with respect to the component α and $L^m(X_j, X_{j+1})$ denotes the space of bounded m-linear maps from X_j to X_{j+1}.

(A4) *Continuity in external parameter:* For each $j \in \mathbb{N}_0$, suppose that the parameters of $\bar{\varphi}_j$ are continuous maps from a metric space M_{ext} of external parameters into $\mathbb{R}$, and that ψ_j, ρ_j also have continuous dependence on M_{ext}. The assumptions (A1–A3) hold uniformly in M_{ext}.

The bounds (1.11) guarantee that Φ is a third-order perturbation in the V component. Moreover, since $\kappa < 1$, the first inequality of (1.10) ensures that the K-direction is contractive for Φ. The assumption that Φ_j is defined on all of X_j is more than needed.

In fact, only the behavior of Φ_j in a small neighbourhood is important, and a weakened version of (A3) is given below as (A3').

Our applications include situations in which β_j approaches a positive limit as $j \to \infty$, but also situations in which β_j is approximately constant in j over a long initial interval $j \leq j_\Omega$ and then abruptly decays to zero. In this setting, j_Ω plays the role of a *mass scale*, due to the fact that in [4] the parameters of $\bar{\varphi}_j$ and the maps ψ_j, ρ_j are continuous functions of a *mass* parameter $m \geq 0$, and $j_\Omega \to \infty$ as $m \downarrow 0$. We are interested in continuity of the flow Φ in this massless limit, and (A4) is designed to accommodate this aspect.

Our main result is the following theorem. In its statement, and throughout the paper, constants in estimates are permitted to depend on c, λ, the constant in the $O(\chi_j)$ bound of (A2), κ, Ω, M, but not on j_Ω.

Theorem 1.2. *Assume (A1–A3) with $\kappa < 1$ and $\Omega \in (1, \kappa^{-1})$. For any $K_* > 0$, there exists $g_* > 0$ such that when $0 < g_0 \leq g_*$ and $\|K_0\|_{\mathcal{W}_0} < K_* g_0^3$ the following hold.*

(i) There exists a global flow of Φ with initial condition (K_0, g_0) and final condition $(z_\infty, \mu_\infty) = (0, 0)$. Let $\bar{V} = (\bar{g}, \bar{z}, \bar{\mu})$ denote the unique global flow of $\bar{\varphi}$ with the same boundary conditions. Then

$$\|K_j\|_{\mathcal{W}_j} = O(\chi_j \bar{g}_j^3), \tag{1.14}$$

$$g_j = \bar{g}_j + O(\bar{g}_j^2 |\log \bar{g}_j|), \tag{1.15}$$

$$z_j = \bar{z}_j + O(\chi_j \bar{g}_j^2 |\log \bar{g}_j|) = O(\chi_j \bar{g}_j), \tag{1.16}$$

$$\mu_j = \bar{\mu}_j + O(\chi_j \bar{g}_j^2 |\log \bar{g}_j|) = O(\chi_j \bar{g}_j), \tag{1.17}$$

and the sequence $(K_j, V_j)_{j \in \mathbb{N}_0}$ is the unique solution to (1.6) *which obeys the boundary conditions and the bounds* (1.14)–(1.17) *(with sufficiently small constants depending on g_0).*

(ii) For each $j \in \mathbb{N}_0$, (K_j, V_j) is continuously Fréchet differentiable in the initial condition (K_0, g_0) (as a map from $\mathcal{W}_0 \oplus \mathbb{R}$ to $\mathcal{W}_j \oplus \mathcal{V}$).

(iii) Assume also (A4). Then (K_j, V_j) is continuous in M_{ext}, and (1.14)–(1.17) *hold uniformly in M_{ext}.*

Alternatively, fix $g_0 > 0$ sufficiently small and $K_0 \in \mathcal{W}_0$, and assume (A1–A2) and (A3') below for this (g_0, K_0). Then (i–iii) also hold.

We do not give a proof, but we expect that the error bounds in (1.14)–(1.17) are optimal. Some indication of this can be found in Remark 3.5 below.

Theorem 1.2 is an analogue of a stable manifold theorem for the non-hyperbolic dynamical system defined by (1.5). It is inspired by [3, Theorem 2.16] which however holds only in the hyperbolic setting. Irwin [8] showed that the stable manifold theorem for hyperbolic dynamical systems is a consequence of the implicit function theorem in Banach spaces (see also [11, 12]). Irwin's approach was inspired by Robbin [10], who showed that the local existence theorem for ordinary differential equations is a consequence of the

implicit function theorem. By contrast, in our proof of Theorem 1.2, we directly apply the local existence theorem for ODEs, without explicit mention of the implicit function theorem. This turns out to be advantageous to deal with the lack of hyperbolicity.

It is not necessary that $\bar{g}_j \to 0$, as we have seen in Example 1.1(ii). However, it can be proved that $\bar{g}_j \to 0$ if β_j is eventually bounded below away from zero, so for (β_j) obeying (A1) the failure of $\bar{g}_j \to 0$ can only occur when $j_\Omega < \infty$. Thus, in any case, $\chi_j \bar{g}_j \to 0$ as $j \to \infty$, and therefore (1.14) and (1.16)–(1.17) imply $K_j \to 0$, $z_j \to 0$, $\mu_j \to \infty$. When $j_\Omega < \infty$, the bounds $z_j, \mu_j = O(\chi_j \bar{g}_j)$ of (1.16)–(1.17) show that z_j and μ_j decay exponentially after the Ω-cut-off time j_Ω; we interpret this as indicating that the boundary condition $(z_\infty, \mu_\infty) = (0, 0)$ is essentially achieved already at j_Ω.

Because of its triangularity, an exact analysis of $\bar{\varphi}$ is straightforward: the three equations for g, z, μ can be solved successively and we do this in Section 2 below. Triangularity does not hold for Φ, and we prove that the flow of Φ nevertheless remains close to the flow of $\bar{\varphi}$ in Section 3.

Our choice of $\bar{\varphi}$ in (1.1) has a specific triangular form. One reason for this is that (1.1) accommodates what is required in our application in [2, 4]. A second reason is that additional nonzero terms in $\bar{\varphi}$ can lead to the failure of Theorem 1.2. The condition that β_j is mainly non-negative is important for the sequence $\bar{g}_j$ of (1.7) to remain bounded. The following example shows that for the ζ_j term in the flow of $\bar{z}$, our sign restriction on ζ_j is also important, since positive ζ_j can lead to violation of a conclusion of Theorem 1.2.

Example 1.3. Suppose that $\zeta_j = \theta_j = \beta_j = 1$, that $\rho = 0$, and that $\bar{g}_0 > 0$ is small. For this constant β sequence, $j_\Omega = \infty$ (for any $\Omega > 1$) and hence $\chi_j = 1$ for all j. As in Example 1.1, $\bar{g}_j \sim j^{-1}$. By (1.1) and (1.7),

$$\bar{z}_{j+1} = \bar{z}_j(1 - \bar{g}_j) - \bar{g}_j^2 = \bar{z}_j \frac{\bar{g}_{j+1}}{\bar{g}_j} - \bar{g}_j^2. \tag{1.18}$$

Let $\bar{y}_j = \bar{z}_j / \bar{g}_j$. Since $\bar{g}_j / \bar{g}_{j+1} = (1 - \bar{g}_j)^{-1} \geq 1$, we obtain $\bar{y}_j \geq \bar{y}_{j+1} + \bar{g}_j$ and hence

$$\bar{y}_j \geq \bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l. \tag{1.19}$$

Suppose that $\bar{z}_j = O(\bar{g}_j)$, as in (1.16). Then $\bar{y}_j = O(1)$ and hence by taking the limit $n \to \infty$ we obtain

$$\bar{y}_j \geq \limsup_{n \to \infty} \left(\bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l \right) \geq -C + \sum_{l=j}^{\infty} \bar{g}_l. \tag{1.20}$$

However, since $\bar{g}_j \sim j^{-1}$, the last sum diverges. This contradiction implies that the conclusion $\bar{z}_j = O(\bar{g}_j)$ of (1.16) is impossible.

1.4 Application

A fundamental element in any renormalisation group analysis concerns the flow of local interactions obtained via iteration of a renormalisation group map [13]. Our flow equation (1.5) arises as part of a renormalisation group study of the long-distance behaviour of the

continuous-time weakly self-avoiding walk on the 4-dimensional hypercubic lattice $\mathbb{Z}^4$. An abbreviated account of this is given in [2], and the full account in [4]. The main result of [4] is that the critical two-point function of the continuous-time weakly self-avoiding walk has $|x|^{-2}$ decay in dimension four. Theorem 1.2 is an essential ingredient in proving this result, and the uniformity of (1.14)–(1.17) in the cut-off time (for a given Ω) is needed. In our application, the index j represents an increasingly large length scale, the spaces $\mathcal{W}_j$ have a subtle definition and are of unbounded dimensions, and their j-dependence is an inevitable consequence of applying the renormalisation group to a lattice model.

The flow equation (1.5) is typical of those encountered in the renormalisation group study of field theories with quartic self-interaction, and we expect our analysis to be of wider applicability, e.g., to the ϕ^4 field theory studied by a different method in [5]. Moreover, we expect our analysis to provide the basis for work in progress which attempts to prove the existence of logarithmic corrections to scaling for the 4-dimensional weakly self-avoiding walk; this was done by a different method for the ϕ^4 field theory in dimension 4 in [6, 7].

1.5 The weaker assumption (A3')

The particular map Φ_j constructed in [4] is singular at $g_j = 0$, and cannot be defined on all of X_j. However, the conclusion of Theorem 1.2 only refers to a small region in X_j, and it is in fact sufficient that Φ_j be defined in that region. This is accommodated by the weaker but more technical assumption (A3') that we discuss next.

Let $g_0 > 0$ be small and let $\bar{V}$ be the unique flow of $\bar{\varphi}$ with initial condition g_0 and with $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$. The existence of this flow is established in Section 2 as part of the proof of Theorem 1.2 (the assumption (A3) plays no role in this). Let $\bar{K}$ be defined inductively by $\bar{K}_0 = K_0$ and $\bar{K}_{j+1} = \psi_j(\bar{x}_j)$, and let $\bar{x} = (\bar{K}, \bar{V}) = (\bar{x}_\alpha)_{\alpha=K,g,z,\mu}$. Given positive constants h_α, we define

$$\mathsf{w}_{\alpha,j} = \mathsf{h}_\alpha \times \begin{cases} \bar{g}_j^3 \chi_j & \alpha = K \\ \bar{g}_j^2 |\log \bar{g}_j| & \alpha = g \\ \bar{g}_j^2 |\log \bar{g}_j| \chi_j & \alpha = z, \mu, \end{cases} \tag{1.21}$$

and then define the subset $D_j \subset X_j$ by

$$D_j = \big\{ x_j \in X_j : x_{\alpha,j} - \bar{x}_{\alpha,j} \in [-\mathsf{w}_{\alpha,j}, \mathsf{w}_{\alpha,j}] \big\}. \tag{1.22}$$

Note that $0 \notin D_j$ for small $\bar{g}_j$. As stated in Theorem 1.2, the conclusion of the theorem continues to hold when (A3) is replaced by the following weaker assumption (A3').

(A3') *The perturbation (weaker version):* There is $M > 0$ such that the maps ψ_j and ρ_j are defined and are three times continuously Fréchet differentiable on D_j of (1.22) with $\mathsf{h}_z, \mathsf{h}_\mu \gg \mathsf{h}_g \gg \mathsf{h}_K \gg 1 + M$, and there exist $\tilde{x}_j \in D_j$ such that

$$\psi_j(\tilde{x}_j) \le M \chi_j \bar{g}_j^3, \quad \rho_j(\tilde{x}_j) \le M \chi_j \bar{g}_j^3. \tag{1.23}$$

The bounds (1.10)–(1.13) on the derivatives hold on D_j.

The assumption (1.23) in (A3') replaces the assumptions $\psi_j(0) = 0$ and $\rho_j(0) = 0$ in (A3), which do need replacement since we no longer assume that ψ_j or ρ_j is defined at 0. It is a generalisation since (1.23) is a consequence of $\psi_j(0) = 0$ and $\rho_j(0) = 0$ along with (1.10)–(1.13); see Lemma 3.4 and (3.98).

In proving Theorem 1.2 in Section 3, we assume that either (A3) or (A3') holds, and comment on the minor differences between the two cases.

2 The quadratic flow

In this section, we study the unique solution $\bar{V} = (\bar{V}_j) = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ of the quadratic flow

$$\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j) \quad \text{with fixed } \bar{g}_0 \text{ and with } (\bar{z}_\infty, \bar{\mu}_\infty) = (0,0). \tag{2.1}$$

Due to the triangular nature of $\bar{\varphi}$, we can obtain detailed information about the sequence $\bar{V}$.

2.1 Flow of $\bar{g}$

We start with the analysis of the sequence $\bar{g}$, which obeys the recursion

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \qquad \bar{g}_0 = g_0 > 0. \tag{2.2}$$

The following lemma collects the information we will need about $\bar{g}$.

Lemma 2.1. *Assume (A1). The following statements hold.*

(i) If $g_0 > 0$ is sufficiently small (depending on $\|\beta\|_\infty$, c, and Ω, but not on j_Ω), then $\bar{g}_j > 0$ for all j,

$$\bar{g}_j = O(\inf_{k\le j} \bar{g}_k), \quad \textit{and} \quad \bar{g}_j \bar{g}_{j+1}^{-1} = 1 + O(\chi_j \bar{g}_j) = 1 + O(g_0). \tag{2.3}$$

(ii) For $n \ge 1$ and $m \ge 0$, there exists $C_{n,m} > 0$ such that for all $k \ge j \ge 0$,

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le C_{n,m} \begin{cases} |\log \bar{g}_k|^{m+1} & n = 1 \\ \chi_j \bar{g}_j^{n-1} |\log \bar{g}_j|^m & n > 1. \end{cases} \tag{2.4}$$

(iii) (a) For $\gamma \ge 0$ and $j \le l$ (with a constant depending on γ but independent of j and l),

$$\prod_{k=j}^{l} (1 - \gamma \beta_k \bar{g}_k)^{-1} = \Big(\frac{\bar{g}_j}{\bar{g}_{l+1}}\Big)^\gamma (1 + O(\chi_j g_j)). \tag{2.5}$$

(b) For $\zeta_j \le 0$ except for c^{-1} values of $j \le j_\Omega$, $\zeta_j = O(\chi_j)$, and $j \le l$, (with a constant independent of j and l),

$$\prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \le O(1). \tag{2.6}$$

(iv) Suppose that $\bar{g}$ and $\mathring{g}$ each satisfy (2.2)*. Let $\delta > 0$. If $|\mathring{g}_0 - \bar{g}_0| \leq \delta \mathring{g}_0$ then $|\mathring{g}_j - \bar{g}_j| \leq \delta \mathring{g}_j (1 + O(\bar{g}_0))$ for all j.*

Proof. (i) By (2.2),

$$\bar{g}_{j+1} = \bar{g}_j(1 - \beta_j \bar{g}_j). \tag{2.7}$$

Since $\beta_j = O(\chi_j)$, by (2.7) the second statement of (2.3) is a consequence the first, so it suffices to verify the first statement of (2.3). Assume inductively that $\bar{g}_j > 0$ and that $\bar{g}_j = O(\inf_{k \leq j} \bar{g}_k)$. It is then immediate from (2.7) that $\bar{g}_{j+1} > 0$ if g_0 is sufficiently small depending on $\|\beta\|_\infty$, and that $\bar{g}_{j+1} \leq \bar{g}_j$ if $\beta_j \geq 0$. By (A1), there are at most c^{-1} values of $j \leq j_\Omega$ for which $\beta_j < 0$. Therefore, by choosing g_0 sufficiently small depending on $\|\beta\|_\infty$ and c, it follows that $\bar{g}_j \leq O(\inf_{k \leq j} \bar{g}_k)$ for all $j \leq j_\Omega$ with a constant that is independent of j_Ω.

To advance the inductive hypothesis for $j > j_\Omega$, we use $1 - t \leq e^{-t}$ and $\sum_{l=j_\Omega}^\infty |\beta_l| \leq \sum_{n=1}^\infty \Omega^{-n} = O(1)$ to obtain, for $j \geq k \geq j_\Omega$,

$$\bar{g}_j \leq \bar{g}_k \exp\left[-\sum_{l=k}^{j-1} \beta_l \bar{g}_l\right] \leq \bar{g}_k \exp\left[C\bar{g}_k \sum_{l=k}^{j-1} |\beta_l|\right] \leq O(\bar{g}_k). \tag{2.8}$$

This shows that $\bar{g}_j = O(\inf_{j_\Omega \leq k \leq j} \bar{g}_k)$. However, by the inductive hypothesis, $\bar{g}_{j_\Omega} = O(\inf_{k \leq j_\Omega} \bar{g}_k)$ for $j \leq j_\Omega$, and hence for $j > j_\Omega$ we do have $\bar{g}_j = O(\inf_{k \leq j} \bar{g}_k)$ as claimed. This completes the verification of the first bound of (2.3) and thus, as already noted, also of the second.

(ii) We first show that if $\psi : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l) \bar{g}_l^2 = \int_{\bar{g}_{k+1}}^{\bar{g}_j} \psi(t)\, dt + O\left(\int_{\bar{g}_{k+1}}^{\bar{g}_j} t^2 |\psi'(t)|\, dt\right). \tag{2.9}$$

To prove (2.9), we apply (2.2) to obtain

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l) \bar{g}_l^2 = \sum_{l=j}^{k} \psi(\bar{g}_l)(\bar{g}_l - \bar{g}_{l+1}) = \sum_{l=j}^{k} \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\, dt. \tag{2.10}$$

The integral can be written as

$$\int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\, dt = \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(t)\, dt + \int_{\bar{g}_{l+1}}^{\bar{g}_l} \int_t^{\bar{g}_l} \psi'(s)\, ds\, dt. \tag{2.11}$$

The first term on the right-hand side of (2.9) is then the sum over l of the first term on the right-hand side of (2.11), so it remains to estimate the double integral. By Fubini's theorem,

$$\begin{aligned} \int_{\bar{g}_{l+1}}^{\bar{g}_l} \int_t^{\bar{g}_l} \psi'(s)\, ds\, dt &= \int_{\bar{g}_{l+1}}^{\bar{g}_l} \int_{\bar{g}_{l+1}}^{s} \psi'(s)\, dt\, ds \\ &= \int_{\bar{g}_{l+1}}^{\bar{g}_l} (s - \bar{g}_{l+1}) \psi'(s)\, ds. \end{aligned} \tag{2.12}$$

By (2.2) and (2.3), for $s \in [\bar{g}_{l+1}, \bar{g}_l]$ we have

$$|s - \bar{g}_{l+1}| \le |\bar{g}_l - \bar{g}_{l+1}| = |\beta_l|\bar{g}_l^2 \le (1 + O(\bar{g}_0))|\beta_l|\bar{g}_{l+1}^2 \le O(s^2). \tag{2.13}$$

This permits us to estimate (2.12) and conclude (2.9).

Direct evaluation of the integrals in (2.9) with $\psi(t) = t^{n-2}|\log t|^m$ gives

$$\sum_{l=j}^{k} \beta_l \bar{g}_l^n |\log \bar{g}_l|^m \le C_{n,m} \begin{cases} |\log \bar{g}_{k+1}|^{m+1} & n = 1 \\ \bar{g}_j^{n-1}|\log \bar{g}_j|^m & n > 1. \end{cases} \tag{2.14}$$

To deduce (2.4), we only consider the case $n > 1$, as the case $n = 1$ is similar. Suppose first that $j \le j_\Omega$ (and $j_\Omega < \infty$). Then (A1) implies that

$$\begin{aligned} \sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le c^{-1} \sum_{l=j}^{j_\Omega} \beta_l \bar{g}_l^n |\log \bar{g}_l|^m + \sum_{l=j}^{j_\Omega} 1_{\beta_l < c} \bar{g}_l^n |\log \bar{g}_l|^m \\ + \sum_{l=j_\Omega+1}^{k} \Omega^{-(l-j_\Omega)_+} \bar{g}_l^n |\log \bar{g}_l|^m. \end{aligned} \tag{2.15}$$

By (2.14), the first term is bounded by $O(\bar{g}_j^{n-1}|\log \bar{g}_j|^m)$. The second term obeys the same bound, by (A1) and (2.3), as does the last term due to the exponential decay. This proves (2.4) for the case $j \le j_\Omega$. On the other hand, if $j > j_\Omega$, then again using the exponential decay of χ_l and (2.3), we obtain

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le C\chi_j \bar{g}_j^n |\log \bar{g}_j|^m \le C g_0 \chi_j \bar{g}_j^{n-1} |\log \bar{g}_j|^m. \tag{2.16}$$

This completes the proof of (2.4) for the case $n > 1$.

(iii-a) By Taylor's theorem and (2.2), there exists $r_k = O(\beta_k \bar{g}_k)^2$ such that

$$(1 - \gamma\beta_k\bar{g}_k)^{-1} = (1 - \beta_k\bar{g}_k)^{-\gamma}(1 + r_k) = \left(\frac{\bar{g}_k}{\bar{g}_{k+1}}\right)^{\gamma}(1 + r_k). \tag{2.17}$$

It suffices to show that

$$\Big|\prod_{k=j}^{l}(1 + r_k) - 1\Big| \le O(\bar{g}_j). \tag{2.18}$$

With the bounds $1 + t \le e^t$ and $\beta_k = O(\chi_k) = O(1)$, we obtain

$$\begin{aligned} \Big|\prod_{k=j}^{l}(1 + r_k) - 1\Big| &= \Big|\sum_{k=j}^{l} r_k \prod_{m=k+1}^{l}(1 + r_m)\Big| \\ &\le \sum_{k=j}^{l} O(\chi_k \bar{g}_k^2) \exp\left[\sum_{m=k+1}^{l} O(\chi_m \bar{g}_m^2)\right]. \end{aligned} \tag{2.19}$$

It then follows from (2.4) that the right-hand side is $O(\chi_j \bar{g}_j)$, and the proof is complete.
(iii-b) Since $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$, by (2.3) with $\bar{g}_0$ sufficiently small, $\prod_{k=j}^{l}(1-\zeta_k \bar{g}_k)^{-1} \le O(1)$ for $l \le j_\Omega$, with a constant independent of j_Ω. For $j \ge j_\Omega$, we use $1/(1-x) \le 2e^x$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ to obtain

$$\prod_{k=j}^{l}(1-\zeta_k \bar{g}_k)^{-1} \le 2\exp\left[\sum_{k=j}^{l}\zeta_k \bar{g}_k\right] \le 2\exp\left[C\bar{g}_j \sum_{k=j_\Omega}^{\infty}|\chi_k|\right] \le O(1). \tag{2.20}$$

The bounds for $l \le j_\Omega$ and $j \ge j_\Omega$ together imply (2.6).
(iv) If $|\mathring{g}_j - \bar{g}_j| \le \delta_j \mathring{g}_j$ then by (2.2),

$$|\mathring{g}_{j+1} - \bar{g}_{j+1}| = |\mathring{g}_j - \bar{g}_j|(1-\beta_j(\mathring{g}_j + \bar{g}_j)) \le \delta_{j+1}\mathring{g}_{j+1} \tag{2.21}$$

with

$$\delta_{j+1} = \delta_j \frac{1-\beta_j(\mathring{g}_j+\bar{g}_j)}{1-\beta_j\mathring{g}_j} = \delta_j\Big(1 - \frac{\beta_j \bar{g}_j}{1-\beta_j \mathring{g}_j}\Big). \tag{2.22}$$

In particular, if $\beta_j \ge 0$, then $\delta_{j+1} \le \delta_j$. By (A1), there are at most c^{-1} values of $j \le j_\Omega$ for which $\beta_j < 0$, and hence $\delta_j \le \delta(1+O(\bar{g}_0))$ for $j \le j_\Omega$. The desired estimate therefore holds for $j \le j_\Omega$. For $j \ge l > j_\Omega$, as in (2.8) we have

$$\prod_{k=l}^{j}(1+O(\beta_k \bar{g}_k)) \le \exp\left[O(\bar{g}_l)\sum_{k=l}^{j}\chi_k\right] \le 1 + O(\bar{g}_0), \tag{2.23}$$

and thus the claim remains true also for $j > j_\Omega$. □

2.2 Flow of $\bar{z}$ and $\bar{\mu}$

We now establish the existence of unique solutions to the $\bar{z}$ and $\bar{\mu}$ recursions with boundary conditions $\bar{z}_\infty = \bar{\mu}_\infty = 0$, and obtain estimates on these solutions.

Lemma 2.2. *Assume (A1–A2). If $\bar{g}_0$ is sufficiently small then there exists a unique solution to* (2.1) *obeying $\bar{z}_\infty = \bar{\mu}_\infty = 0$. This solution obeys $\bar{z}_j = O(\chi_j \bar{g}_j)$ and $\bar{\mu}_j = O(\chi_j \bar{g}_j)$.*

Proof. By (1.1), $\bar{z}_{j+1} = \bar{z}_j - \zeta_j \bar{g}_j \bar{z}_j - \theta_j \bar{g}_j^2$, so that

$$\bar{z}_j = \prod_{k=j}^{n}(1-\zeta_k\bar{g}_k)^{-1}\bar{z}_{n+1} + \sum_{l=j}^{n}\prod_{k=j}^{l}(1-\zeta_k\bar{g}_k)^{-1}\theta_l \bar{g}_l^2. \tag{2.24}$$

In view of (2.6), whose assumptions are satisfied by (A2), the unique solution to the recursion for $\bar{z}$ which obeys the boundary condition $\bar{z}_\infty = 0$ is

$$\bar{z}_j = \sum_{l=j}^{\infty}\prod_{k=j}^{l}(1-\zeta_k\bar{g}_k)^{-1}\theta_l \bar{g}_l^2, \tag{2.25}$$

and by (A2) and (2.4),

$$|\bar{z}_j| \leq \sum_{l=j}^{\infty} O(\chi_l)\bar{g}_l^2 \leq O(\chi_j \bar{g}_j). \tag{2.26}$$

For $\bar{\mu}$, we first define

$$\sigma_j = -\eta_j \bar{g}_j - \gamma_j \bar{z}_j - \upsilon_j^{gg} \bar{g}_j^2 - \upsilon_j^{gz} \bar{g}_j \bar{z}_j - \upsilon_j^{zz} \bar{z}_j^2, \quad \tau_j = \upsilon_j^{g\mu} \bar{g}_j + \upsilon_j^{z\mu} \bar{z}_j, \tag{2.27}$$

so that the recursion for $\bar{\mu}$ can be written as

$$\bar{\mu}_{j+1} = (\lambda_j - \tau_j)\bar{\mu}_j + \sigma_j. \tag{2.28}$$

Alternatively,

$$\bar{\mu}_j = (\lambda_j - \tau_j)^{-1}(\bar{\mu}_{j+1} - \sigma_j). \tag{2.29}$$

Given $\alpha \in (\lambda^{-1}, 1)$, we can choose $\bar{g}_0$ sufficiently small that

$$\tfrac{1}{2}\lambda^{-1} \leq (\lambda_j - \tau_j)^{-1} \leq \alpha. \tag{2.30}$$

The limit of repeated iteration of (2.29) gives

$$\bar{\mu}_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \sigma_l \tag{2.31}$$

as the unique solution which obeys the boundary condition $\mu_\infty = 0$. Geometric convergence of the sum is guaranteed by (2.30), together with the fact that $\sigma_j \leq O(\chi_j \bar{g}_j) \leq O(1)$. To estimate (2.31), we use

$$|\bar{\mu}_j| \leq \sum_{l=j}^{\infty} \alpha^{l-j+1} O(\chi_l \bar{g}_l). \tag{2.32}$$

Since $\alpha < 1$, the first bound of (2.3) and monotonicity of χ imply that

$$|\bar{\mu}_j| \leq O(\chi_j \bar{g}_j), \tag{2.33}$$

and the proof is complete. $\square$

2.3 Differentiation of quadratic flow

The following lemma gives estimates on the derivatives of the components of $\bar{V}_j$ with respect to the initial condition $\bar{g}_0$. We write f' for the derivative of f with respect to $g_0 = \bar{g}_0$. These estimates will be an ingredient in the proof of Theorem 1.2(ii).

Lemma 2.3. *For each $j \geq 0$, $\bar{V}_j = (\bar{g}_j,\, \bar{z}_j,\, \bar{\mu}_j)$ is twice differentiable with respect to the initial condition $\bar{g}_0 > 0$, and the derivatives obey*

$$\bar{g}_j' = O\Big(\frac{\bar{g}_j^2}{\bar{g}_0^2}\Big), \qquad \bar{z}_j' = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\Big), \qquad \bar{\mu}_j' = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\Big), \tag{2.34}$$

$$\bar{g}_j'' = O\Big(\frac{\bar{g}_j^2}{\bar{g}_0}\Big), \qquad \bar{z}_j'' = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\Big), \qquad \bar{\mu}_j'' = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\Big). \tag{2.35}$$

Proof. Differentiation of (1.7) gives

$$\bar{g}'_{j+1} = \bar{g}'_j(1 - 2\beta_j \bar{g}_j), \tag{2.36}$$

from which we conclude by iteration and $\bar{g}'_0 = 1$ that for $j \geq 1$,

$$\bar{g}'_j = \prod_{l=0}^{j-1}(1 - 2\beta_l \bar{g}_l). \tag{2.37}$$

Therefore, by (2.5),

$$\bar{g}'_j = \Big(\frac{\bar{g}_j}{\bar{g}_0}\Big)^2 (1 + O(\bar{g}_0)). \tag{2.38}$$

For the second derivative, we use $\bar{g}''_0 = 0$ and $\bar{g}''_{j+1} = \bar{g}''_j(1 - 2\beta_j \bar{g}_j) - 2\beta_j \bar{g}'^2_j$ to obtain

$$\bar{g}''_j = -2 \sum_{l=0}^{j-1} \beta_l \bar{g}'^2_l \prod_{k=l}^{j-2}(1 - 2\beta_k \bar{g}_k). \tag{2.39}$$

With the bounds of Lemma 2.1, this gives

$$\bar{g}''_j = O\Big(\frac{\bar{g}_j}{\bar{g}_0}\Big)^2 \sum_{l=0}^{j-1} \beta_l \bar{g}_l^2 = O\Big(\frac{\bar{g}_j^2}{\bar{g}_0}\Big). \tag{2.40}$$

For $\bar{z}$, we define $\sigma_{j,l} = \prod_{k=j}^{l}(1 - \zeta_k \bar{g}_k)^{-1}$. Then (2.25) becomes $\bar{z}_j = \sum_{l=j}^{\infty} \sigma_{j,l} \theta_l \bar{g}_l^2$. By (2.6), $\sigma_{j,l} = O(1)$. It then follows from (A2), (2.38), and Lemma 2.1(ii,iii-b) that

$$\sigma'_{j,l} = \sigma_{j,l} \sum_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \zeta_k \bar{g}'_k = \sum_{k=j}^{l} O(\zeta_k \bar{g}'_k) = O\Big(\chi_j \frac{\bar{g}_j}{\bar{g}_0^2}\Big). \tag{2.41}$$

We differentiate (2.25) and apply (2.38) and Lemma 2.1(ii) to obtain

$$\bar{z}'_j = \sum_{l=j}^{\infty} \sigma'_{j,l} \theta_l \bar{g}_l^2 + 2 \sum_{l=j}^{\infty} \sigma_{j,l} \theta_l \bar{g}_l \bar{g}'_l = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\Big). \tag{2.42}$$

Similarly, $\sigma''_{j,l} = O(\bar{g}_j^2/\bar{g}_0^4)$ and

$$\bar{z}''_j = \sum_{l=j}^{\infty} \sigma''_{j,l} \theta_l \bar{g}_l^2 + 4 \sum_{l=j}^{\infty} \sigma'_{j,l} \theta_l \bar{g}_l \bar{g}'_l + 2 \sum_{l=j}^{\infty} \sigma_{j,l} \theta_l (\bar{g}_l \bar{g}''_l + \bar{g}'^2_l) = O\Big(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\Big) \tag{2.43}$$

using the fact that $\bar{g}_j^3/\bar{g}_0^4 = O(\bar{g}_j^2/\bar{g}_0^3)$ by (2.3). It is straightforward to justify the differentiation under the sum in (2.42)–(2.43).

For $\bar{\mu}_j$, we recall from (2.30)–(2.31) that

$$\bar{\mu}_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \sigma_l, \tag{2.44}$$

with τ_j and σ_l given by (2.27), and with $0 \le (\lambda_j - \tau_j)^{-1} \le \alpha < 1$. This gives

$$\bar{\mu}'_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \left(\sigma'_l + \sum_{i=j}^{l} (\lambda_i - \tau_i)^{-1} \tau'_i \right). \tag{2.45}$$

The first product is bounded by α^{l-j+1}, and this exponential decay, together with (2.27), (2.26), and the bounds just proved for $\bar{g}'$ and $\bar{z}'$, lead to the upper bound $|\bar{\mu}'_j| \le O(\chi_j \bar{g}_j^2 \bar{g}_0^{-2})$ claimed in (2.34). Straightforward further calculation leads to the bound on $\bar{\mu}''_j$ claimed in (2.35) (the leading behaviour can be seen from the $\bar{z}''_j$ contribution to the σ''_l term). $\square$

3 Proof of main result

In this section, we prove Theorem 1.2. We begin in Section 3.1 with a sketch of the main ideas, without entering into details. The remainder of Section 3 expands the sketch into a complete proof.

3.1 Proof strategy

Two difficulties in proving Theorem 1.2 are: (i) from the point of view of dynamical systems, the evolution map Φ is not hyperbolic; and (ii) from the point of view of nonlinear differential equations, a priori bounds that any solution to (1.6) must satisfy are not readily available due to the presence of both initial and final boundary conditions.

Our strategy is to consider the one-parameter family of evolution maps $(\Phi^t)_{t\in[0,1]}$ defined by

$$\Phi^t(x) = \Phi(t,x) = (\psi(x), \bar{\varphi}(x) + t\rho(x)) \quad \text{for } t \in [0,1], \tag{3.1}$$

with the t-independent boundary conditions that (K_0, g_0) is given and that $(z_\infty, \mu_\infty) = (0,0)$. This family interpolates between the problem $\Phi^1 = \Phi$ we are interested in, and the simpler problem $\Phi^0 = \bar{\Phi} = (\psi, \bar{\varphi})$. The unique solution for $\bar{\Phi}$ is $\bar{x}_j = (\bar{K}_j, \bar{V}_j)$, where $\bar{V}$ is the unique solution of $\bar{\varphi}$ from Section 2, and where $\bar{K}_j$ is defined inductively for $j \ge 0$ by

$$\bar{K}_{j+1} = \psi_j(\bar{V}_j, \bar{K}_j), \qquad \bar{K}_0 = K_0. \tag{3.2}$$

We refer to $\bar{x}$ as the *approximate flow*.

We seek a t-dependent global flow x which obeys the generalisation of (1.6) given by

$$x_{j+1} = \Phi_j^t(x_j). \tag{3.3}$$

Assuming that $x_j = x_j(t)$ is differentiable in t for each $j \in \mathbb{N}_0$, we set

$$\dot{x}_j = \frac{\partial}{\partial t} x_j. \tag{3.4}$$

Then differentiation of (3.3) shows that a family of flows $x = (x_j(t))_{j\in\mathbb{N}_0, t\in[0,1]}$ must satisfy the infinite nonlinear system of ODEs

$$\dot{x}_{j+1} - D_x\Phi_j(t,x_j)\dot{x}_j = \rho_j(x_j), \quad x_j(0) = \bar{x}_j. \tag{3.5}$$

Conversely, any solution $x(t)$ to (3.5), for which each x_j is continuously differentiable in t, gives a global flow for each Φ^t.

We claim that (3.5) can be reformulated as a well-posed nonlinear ODE

$$\dot{x} = F(t,x), \quad x(0) = \bar{x}, \tag{3.6}$$

in a Banach space of sequences $x = (x_0, x_1, \dots)$ with carefully chosen weights, and for a suitable nonlinear functional F. To see this, consider the *linear* equation

$$y_{j+1} - D_x\Phi_j(t,x_j)y_j = r_j, \tag{3.7}$$

where the sequences x and r are held fixed. Its solution with the same boundary conditions as stated below (3.1) is written as $y = S(t,x)r$. Then we define F, which we consider as a map on sequences, by

$$F(t,x) = S(t,x)\rho(x). \tag{3.8}$$

Thus $y = F(t,x)$ obeys the equation $y_{j+1} - D_x\Phi_j(t,x_j)y_j = \rho_j(x)$, and hence (3.6) is equivalent to (3.5) with the same boundary conditions.

The main work in the proof is to obtain good estimates for $S(t,x)$, in the Banach space of weighted sequences, which allow us to treat (3.6) by the standard theory of ODE. We establish bounds on the solution simultaneously with existence, via the weights in the norm. These weights are useful to obtain bounds on the solution, but they are also essential in the formulation of the problem as a well-posed ODE.

As we will see in more detail in Section 3.4.1 below, the occurrence of $D_x\Phi_j(t,x_j)$ in (3.5), rather than the naive linearisation $D_x\Phi_j(0)$, replaces the eigenvalue 1 in the upper left corner of the square matrix in (1.1) by a smaller eigenvalue $1 - 2\beta_j g_j$. This helps address difficulty (i) mentioned above. Also, the weights guarantee that a solution in the Banach space obeys the final conditions $(z_\infty, \mu_\infty) = (0,0)$, thereby helping to solve difficulty (ii).

3.2 Sequence spaces and weights

We now introduce the Banach spaces of sequences used in the reformulation of (3.5) as an ODE. These are weighted l^∞-spaces.

Definition 3.1. Let X^* be the space of sequences $x = (x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$, with the topology of component-wise convergence: $x^n \to x$ in X^* if $x^n_j \to x_j$ in X_j for each $j \in \mathbb{N}_0$.

Definition 3.2. For each $\alpha = K, g, z, \mu$ and $j \in \mathbb{N}_0$, we fix a positive weight $w_{\alpha,j} > 0$. We write $x_j \in X_j = \mathcal{W}_j \oplus \mathcal{V}$ as $x_j = (x_{\alpha,j})_{\alpha=K,g,z,\mu}$. Let

$$\|x_j\|_{X^w_j} = \max_{\alpha=K,g,z,\mu} (w_{\alpha,j})^{-1}\|x_{\alpha,j}\|_{X_j}, \quad \|x\|_{X^w} = \sup_{j\in\mathbb{N}_0} \|x_j\|_{X^w_j}, \tag{3.9}$$

$$X^w = \{x \in X^* : \|x\|_{X^w} < \infty\}. \tag{3.10}$$

Definition 3.3. Let $BX^* = C(M_{\text{ext}}, X^*)$. Given continuous weight functions $w_{\alpha,j} : M_{\text{ext}} \to \mathbb{R}_+$, let

$$\|x\|_{BX^w} = \sup_{m\in M_{\text{ext}}} \|x(m)\|_{X^{w(m)}}, \quad BX^w = \{x \in BX^* : \|x\|_{BX^w} < \infty\}.$$

It is not difficult to check that X^w and BX^w are Banach spaces. Different choices of weights w will be needed. These are all defined in terms of the sequence $\mathring{g} = (\mathring{g}_j)_{j\in\mathbb{N}_0}$ which is the same as the sequence $\bar{g}$ for a *fixed* $\mathring{g}_0$; i.e., given $\mathring{g}_0 > 0$, it satisfies $\mathring{g}_{j+1} = \mathring{g}_j - \beta_j \mathring{g}_j^2$. We will assume that $g_0 = \bar{g}_0$ is close to $\mathring{g}_0$, but not necessarily that they are equal. Given constants $\mathsf{h}_\alpha > 0$ for $\alpha = K, g, z, \mu$, we define the weights $\mathsf{w} = \mathsf{w}(\mathsf{h}, \mathring{g})$ and $\mathsf{r} = \mathsf{r}(\mathsf{h}, \mathring{g})$ by

$$\mathsf{w}_{\alpha,j} = \mathsf{h}_\alpha \times \begin{cases} \mathring{g}_j^3 \chi_j & \alpha = K \\ \mathring{g}_j^2 |\log \mathring{g}_j| & \alpha = g \\ \mathring{g}_j^2 |\log \mathring{g}_j| \chi_j & \alpha = z, \mu, \end{cases} \qquad \mathsf{r}_{\alpha,j} = \mathsf{h}_\alpha \mathring{g}_j^3 \chi_j, \tag{3.11}$$

where (χ_j) is the Ω-dependent sequence defined by (1.9). The use of $\mathring{g}$ permits us to vary the initial condition g_0 without changing the Banach spaces $X^{\mathsf{w}}, X^{\mathsf{r}}$. Note that the weight $\mathsf{w}_{g,j}$ does not include a factor χ_j, and thus does not go to zero when $\mathring{g}_j$ does not go to zero (as e.g. in Example 1.1(ii)).

Let $\bar{x} = (\bar{K}, \bar{V}) = \bar{x}(K_0, g_0)$ denote the sequence in X^* that is uniquely determined from the boundary conditions $(\bar{K}_0, \bar{g}_0) = (K_0, g_0)$ and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$ via $\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j)$ and $\bar{K}_{j+1} = \psi_j(\bar{K}_j, \bar{V}_j)$, whenever the latter is well-defined. Given an initial condition $(\mathring{K}_0, \mathring{g}_0)$, let $\mathring{x} = \bar{x}(\mathring{K}_0, \mathring{g}_0)$. We need estimates on $\mathring{x}$. It has been shown already in Lemma 2.2 that $\mathring{z}_j, \mathring{\mu}_j = O(\chi_j \mathring{g}_j)$. The following lemma establishes an estimate for $\|Kr_j\|_{\mathcal{W}_j}$ under assumptions (A1–A2) and (A3) or (A3'). Its proof is straightforward, but we defer it to Section 3.5 where we also prove a related statement.

Lemma 3.4. *Assume (A1–A2), (A3) or (A3'), and that $\mathring{g}_0 > 0$ is sufficiently small. Then (A3'), or $\|\mathring{K}_0\|_{\mathcal{W}_0} \le O(\mathring{g}_0^3)$ and (A3), imply $\|\mathring{K}_j\|_{\mathcal{W}_j} \le O(\chi_j \mathring{g}_j^3)$ (with different constants in the two bounds).*

Let B denote the closed unit ball in X^{w}. The proof of Theorem 1.2 uses the affine space $\mathring{x} + \mathsf{B}$. Note that under (A3'), $\mathring{x} + \mathsf{B} = \{x \in X^* : x_j \in D_j\}$, so that Φ is well-defined on $\mathring{x} + \mathsf{B}$.

Remark 3.5. The weights w apply to the sequence $\dot{x}$. As motivation for their definition, consider the explicit example of $\rho_j(x_j) = \chi_j g_j^3$. In this case, the g equation becomes simply

$$g_{j+1} = g_j - \beta_j g_j^2 + t\chi_j g_j^3. \tag{3.12}$$

With the notation $\dot{g}_j = \frac{\partial}{\partial t} g_j^t$, differentiation gives

$$\dot{g}_{j+1} = \dot{g}_j (1 - 2\beta_j g_j + 3t\chi_j g_j^2) + \chi_j g_j^3. \tag{3.13}$$

Thus, by iteration, using $\dot{g}_0 = 0$, we obtain

$$\dot{g}_j = \sum_{l=0}^{j-1} \chi_l g_l^3 \prod_{k=l+1}^{j-1} (1 - 2\beta_k g_k + 3t\chi_k g_k^3). \tag{3.14}$$

For simplicity, consider the case $t = 0$, for which $g = \bar{g}$. In this case, it follows from (2.5), (2.3), and (2.14) that

$$\dot{g}_j \le O(1) \sum_{l=0}^{j-1} \left(\frac{\bar{g}_j}{\bar{g}_{l+1}}\right)^2 \chi_l \bar{g}_l^3 = O(1) g_j^2 \sum_{l=0}^{j-1} \chi_l \bar{g}_l \le O(1) \bar{g}_j^2 |\log \bar{g}_j|, \tag{3.15}$$

which produces the weight $\mathsf{w}_{g,j}$ of (3.11). (It can be verified using (2.9) that if we replace χ_j by β_j in the above then no smaller weight will work.)

3.3 Reduction to a linear equation with nonlinear perturbation

For given sequences $x, r \in X^*$, we now consider the equation

$$y_{j+1} - D_x\Phi_j(t, x_j)y_j = r_j. \tag{3.16}$$

With x and r fixed, this is an inhomogeneous linear equation in y. Lemma 3.6 below, which lies at the heart of the proof of Theorem 1.2, obtains bounds on solutions to (3.16), including bounds on its x-dependence. The latter will allow us to use the standard theory of ODE in Banach spaces to treat the original nonlinear equation, where x and r are both functionals of the solution y, as a perturbation of the linear equation.

It is convenient to make the decomposition $X_j = E_j \oplus F_j$ with $E_j = \mathcal{W}_j \oplus \mathbb{R}$ and $F_j = \mathbb{R} \oplus \mathbb{R}$, for which we write $x = (u, v)$ with $u = (K, g)$ and $v = (z, \mu)$. We denote by π_α the projection operator onto the α-component of the space in which it is applied, where α can be in any of $\{K, V\}$, $\{u, v\} = \{(K, g), (z, \mu)\}$, or $\{K, g, z, \mu\}$.

Recall that the spaces of sequences X^w are defined in Definition 3.2 and the specific weights w and r in (3.11).

Lemma 3.6. *Assume (A1–A3) and let $\varepsilon > 0$. If $\mathsf{h}_z, \mathsf{h}_\mu \gg \mathsf{h}_g$, then there exists $C > 0$ (independent of ε), such that for all positive $\mathring{g} \ll 1$ the following hold for all $t \in [0,1]$, $x \in \mathring{x} + \mathsf{B}$.*

(i) For $r \in X^{\mathsf{r}}$, there exists a unique solution $y = S(t,x)r \in X^{\mathsf{w}}$ of (3.16) with boundary conditions $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$.

(ii) The linear solution operator $S(t,x)$ satisfies

$$\|S(t,x)\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \le C. \tag{3.17}$$

(iii) $S : [0,1] \times (\mathring{x} + \mathsf{B}) \to L(X^{\mathsf{w}}, X^{\mathsf{r}})$ is continuously Fréchet differentiable, with

$$\|D_x S(t,x)\|_{L(X^{\mathsf{w}}, L(X^{\mathsf{r}}, X^{\mathsf{w}}))} \le \varepsilon. \tag{3.18}$$

Moreover, if (A4) holds then the analogous statements hold in the B-spaces, upon replacing all X^w by BX^w.
Given $\mathring{g}_0 > 0$ sufficiently small and $\mathring{K}_0 \in \mathcal{W}_0$, (i–iii) continue to hold when (A3) is replaced by (A3') defined via $(K_0, g_0) = (\mathring{K}_0, \mathring{g}_0)$.

Lemma 3.6 needs to be supplemented with information about the initial condition $\bar{x}$ and the perturbation ρ for the analysis of (3.6) with (3.8). (Note that $\bar{x}$ is a sequence serving as initial condition for the ODE (3.5), not an initial condition for the flow equation (1.5).) Some information about $\bar{x}$ is already contained in Lemmas 2.2 and 3.4. For ρ, we define $\rho : \mathring{x} + \mathsf{B} \to X^*$ by

$$(\rho(x))_0 = 0, \quad (\rho(x))_{j+1} = \rho_j(x_j), \tag{3.19}$$

where ρ_j is the map of (1.5). The map $\psi : \mathring{x} + \mathsf{B} \to X^*$ is defined similarly. The next lemma derives straightforward consequences of the assumption (A3) or (A3') for ρ and ψ in terms of the weighted spaces. Although the proof of Theorem 1.2 only directly requires the estimates for ρ, we will also need bounds on ψ to prove Lemma 3.6, so for convenience we combine both in a single lemma.

Lemma 3.7. *Assume (A3) or (A3') and let $\varepsilon > 0$ and $\omega > \kappa\Omega$. If $\mathsf{h}_g, \mathsf{h}_z, \mathsf{h}_\mu \gg \mathsf{h}_K \gg 1$ and $\mathring{g}_0 \ll 1$, then $\rho : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ and $\psi : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ are each twice continuously Fréchet differentiable, and*

$$\|\rho(x)\|_{X^{\mathsf{r}}} \leq \varepsilon, \quad \|D_x\rho(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \leq \varepsilon, \quad \|D_x^2\rho(x)\|_{L^2(X^{\mathsf{w}},X^{\mathsf{r}})} \leq \varepsilon, \tag{3.20}$$

$$\|D_x\psi(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \leq \omega, \quad \|D_x^2\psi(x)\|_{L^2(X^{\mathsf{w}},X^{\mathsf{r}})} \leq \varepsilon. \tag{3.21}$$

Moreover, if (A4) holds then the analogous statements hold in the B-spaces, upon replacing all X^w by BX^w.

We defer the proofs of Lemmas 3.6–3.7 to Sections 3.4 and 3.6, respectively. Given these, and given Lemma 3.4, we now prove Theorem 1.2(i).

Proof of Theorem 1.2(i). For $t \in [0,1]$ and $x \in \mathring{x} + \mathsf{B}$, let

$$F(t,x) = S(t,x)\rho(x). \tag{3.22}$$

Given $\varepsilon > 0$, it follows from Lemmas 3.6–3.7 that if $\mathsf{h}_z, \mathsf{h}_\mu \gg \mathsf{h}_g \gg \mathsf{h}_K \gg 1$, and if $\mathring{g}_0 > 0$ is sufficiently small, then $F : [0,1] \times (\mathring{x} + \mathsf{B}) \to X^{\mathsf{w}}$ is continuously Fréchet differentiable, and

$$\begin{aligned} \|D_xF(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \leq \|[D_xS(t,x)]\rho(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \\ + \|S(t,x)[D_x\rho(x)]\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \leq C\varepsilon. \end{aligned} \tag{3.23}$$

We choose ε sufficiently small that $\theta = C\varepsilon < 1$. Then $\|D_xF(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \leq \theta < 1$. Similarly,

$$\|F(t,x)\|_{L(X^{\mathsf{w}})} \leq \theta < 1. \tag{3.24}$$

For $y \in \mathsf{B}$, let

$$\mathring{F}(t,y) = F(t, \mathring{x} + y). \tag{3.25}$$

Let $X_0^{\mathsf{w}} = \{y \in X^{\mathsf{w}} : \pi_u y_0 = 0\}$ and $\mathsf{B}_0 = \mathsf{B} \cap X_0^{\mathsf{w}}$. Then by the statement about boundary conditions of Lemma 3.6(i), and by (3.24), $\mathring{F}(t, \mathsf{B}_0) \subseteq \theta\mathsf{B}_0$. With (3.23)–(3.24), the standard local existence theory for ODEs on Banach spaces [1, Lemma 1] implies that the initial value problem

$$\dot{y} = \mathring{F}(t,y), \quad y(0) = 0 \tag{3.26}$$

has a unique solution $y : [0,1] \to X_0^{\mathsf{w}}$ in C^1 with $y(t) \in \theta\mathsf{B}_0$ for all $t \in [0,1]$.

In particular, as discussed around (3.6), it follows that $x = \mathring{x} + y(1)$ is a solution to (1.6). By construction, $\pi_u x_0 = \pi_u \mathring{x}_0 = (\mathring{K}_0, \mathring{g}_0)$, and we may choose $(\mathring{K}_0, \mathring{g}_0)$ to be any given small (K_0, g_0). Also, $\pi_v y_\infty(1) = 0$ because $y(1) \in X^{\mathsf{w}}$, and since $\pi_v \mathring{x}_\infty = 0$, it is also true that $\pi_v x_\infty = 0$. Thus x satisfies the required boundary conditions. The estimates

(1.14)–(1.17) are an immediate consequence of the bounds on $\mathring{x}$ of Lemmas 2.2 and 3.4 together with the fact that $y \in X^{\mathsf{w}}$ with the weights (3.11).

Conversely, given a solution x to (1.6) with $x - \mathring{x} \in \mathsf{B}$, we may construct a solution $x(t)$ to $\dot{x} = F(x,t)$ with $x(1) = x$ for $t \in [0,1]$ by considering the ODE backwards in time, which is equally well-posed, and the uniqueness of such solutions implies that x coincides with the corresponding solution constructed by starting at $x(0) = \mathring{x}(g_0)$. This completes the proof of Theorem 1.2(i). $\square$

To prove (ii) of Theorem 1.2, we need to know that $\bar{x}$ is differentiable in the space $\mathring{x} + \mathsf{B}$. The smoothness of $\bar{x}$ is addressed in the following lemma, whose proof is deferred to Section 3.5.

Lemma 3.8. *Assume (A1–A2), and (A3) or (A3'). Assume also $\mathring{g}_0 \ll 1$. For any $\theta \in [0,1)$, there exists a neighbourhood $\bar{\mathsf{I}} = \bar{\mathsf{I}}_\theta \subset \mathcal{W}_0 \oplus \mathbb{R}_+$ of $(\mathring{K}_0, \mathring{g}_0)$ such that $\bar{x} : \bar{\mathsf{I}} \to \mathring{x} + (1-\theta)\mathsf{B}$ is continuously Fréchet differentiable. Moreover, if (A4) holds then the analogous statement holds in the B-spaces, upon replacing X^{w} by BX^{w}.*

Proof of Theorem 1.2(ii). Fix an initial condition (K_0, g_0) obeying the hypothesis of Theorem 1.2(i), and let $\bar{x}$ be the corresponding approximate flow. Let $\mathring{g}_0 = g_0$ and let $\bar{\mathsf{I}}$ be the neighbourhood of (K_0, g_0) defined by Lemma 3.8 in terms of some $\theta < 1$ as below (3.23). By Lemma 3.8, $\bar{x} : \bar{\mathsf{I}} \to X^{\mathsf{w}}$ is Fréchet differentiable. Then the map $\bar{F} : [0,1] \times \bar{\mathsf{I}} \times \theta \mathsf{B}_0 \to X_0^{\mathsf{w}}$ given by

$$\bar{F}(t, u_0, y) = F(t, \bar{x}(u_0) + y) \tag{3.27}$$

is Fréchet differentiable. It follows from [9, Theorem 5.2] that

$$\dot{y} = \bar{F}(t, u_0, y), \quad y(0) = 0 \tag{3.28}$$

has a unique solution $y : [0,1] \times \bar{\mathsf{I}} \to X_0^{\mathsf{w}}$, and that y is in C^1. In particular, this implies that, as an element of X_j, $x_j = (K_j, V_j)$ is a continuously Fréchet differentiable function of (K_0, g_0). This completes the proof of Theorem 1.2(ii). $\square$

The continuity statement of Theorem 1.2(iii) cannot be formulated in terms of the spaces X^{w}, as was the case for Theorem 1.2(i–ii), since the weights (3.11) depend on $m \in M_{\text{ext}}$ through $\mathring{g}$. The remedy is to consider the larger space BX^{w} of sequences of functions of m with weights that vary both in j and m.

Proof of Theorem 1.2(iii). To deduce continuity in X^* of the solution as a function of the external parameter $m \in M_{\text{ext}}$, we can repeat the same argument with all X^w spaces replaced by the corresponding BX^w. Then all statements remain true without additional change, by the uniformity assumption of (A4). Continuity follows because $BX^{\mathsf{w}} \subset C(M_{\text{ext}}, X^*)$. This completes the proof of Theorem 1.2(iii). $\square$

It now remains only to prove Lemmas 3.4 and 3.6–3.8. We begin with Lemma 3.6, which lies at the heart of the proof.

3.4 Proof of Lemma 3.6

The proof proceeds in three steps. The first two steps concern an approximate version of the equation and the solution of the approximate equation, and the third step treats (3.16) as a small perturbation of this approximation.

3.4.1 Step 1. Approximation of the linear equation

We extend $\bar{\varphi}$ to X_j by making it act trivially on the K-component, i.e., let $\bar{\varphi}^0 = 0 \oplus \bar{\varphi}$. Explicit computation of the derivative of $\bar{\varphi}_j$ of (1.5), using (1.1), shows that

$$D\bar{\varphi}^0_j(x_j) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1-2\beta_j g_j & 0 & 0 \\ \hline 0 & -\tilde{\xi}_j & 1-2\zeta_j g_j & 0 \\ 0 & \tilde{\eta}_j & -\tilde{\gamma}_j & \tilde{\lambda}_j \end{array}\right), \tag{3.29}$$

with

$$\begin{aligned} \tilde{\eta}_j &= \eta_j - 2\upsilon_j^{gg} g_j - \upsilon_j^{gz} z_j - \upsilon_j^{g\mu}\mu_j, \\ \tilde{\gamma}_j &= \gamma_j - \upsilon_j^{gz} g_j - 2\upsilon_j^{zz} z_j - \upsilon_j^{z\mu}\mu_j, \\ \tilde{\lambda}_j &= \lambda_j - \upsilon_j^{g\mu} g_j - \upsilon_j^{z\mu} z_j, \\ \tilde{\xi}_j &= 2\theta_j g_j + 2\zeta_j z_j. \end{aligned} \tag{3.30}$$

The block matrix structure in (3.29) is with respect to the decomposition $X_j = E_j \oplus F_j$ introduced in Section 3.3.

The matrix $D\bar{\varphi}^0_j(x_j)$ depends on $x_j \in X_j$, of course, but it is convenient to approximate it by

$$L_j = D\bar{\varphi}^0_j(\mathring{x}_j) = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}, \tag{3.31}$$

where the blocks A_j, B_j, and C_j of L_j are defined by evaluating the blocks of the matrix (3.29) at $\mathring{x}_j$ rather than at x_j (given explicitly in (3.37) below). We will study the equation

$$y_{j+1} = L_j y_j + r_j, \tag{3.32}$$

which approximates (3.16). Lemma 3.9 below gives a useful reformulation of (3.32). For its statement, we define linear operators $H : D(H) \to X^*$ and $U : D(U) \to X^*$ (where $D(H)$ and $D(U)$ are the subspaces of X^* on which the infinite sums converge) by

$$\pi_u H = 0, \quad (\pi_v H x)_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} B_l \pi_u x_l, \tag{3.33}$$

and

$$\begin{aligned} (\pi_u U x)_j &= \sum_{l=0}^{j-1} A_{j-1} \cdots A_{l+1} \pi_u x_l, \\ (\pi_v U x)_j &= -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} \pi_v x_l. \end{aligned} \tag{3.34}$$

It follows from the definitions (recalling $\pi_K A_j = 0 = A_j \pi_K$) that

$$\pi_K H = 0 = H\pi_K, \quad \pi_V H = H = H\pi_V, \quad \pi_K U = U\pi_K, \quad \pi_V U = U\pi_V. \tag{3.35}$$

The empty product in the formula for $\pi_u Ux$ is interpreted as the identity, so the term in the sum corresponding to $l = j-1$ is simply $\pi_u x_j$. It is thus not the case that $\pi_K U = 0$, since $\pi_K x_j$ is not assumed to vanish.

Lemma 3.9. *Assume (A1–A2) and that $\mathring{g}_0 > 0$ is sufficiently small. If $y \in X^*$ satisfies $\pi_u y_0 = 0$ and $\pi_v y_\infty = 0$, then (3.32) is equivalent to*

$$y = Hy + Ur. \tag{3.36}$$

The proof is straightforward, but requires an estimate on the product of the matrices C_j which we will prove first. Products of the C_j and A_j will also play an important role in the analysis of the operators H and U in the following section, so that it is convenient to prove a more precise statement about them now than what is needed for the proof of Lemma 3.9. Let us first record explicitly the blocks of L_j:

$$A_j = \begin{pmatrix} 0 & 0 \\ 0 & 1-2\beta_j\mathring{g}_j \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & -\mathring{\xi}_j \\ 0 & \mathring{\eta}_j \end{pmatrix}, \quad C_j = \begin{pmatrix} 1-2\zeta_j\mathring{g}_j & 0 \\ -\mathring{\gamma}_j & \mathring{\lambda}_j \end{pmatrix} \tag{3.37}$$

with $\mathring{\eta}_j$, $\mathring{\gamma}_j$, $\mathring{\lambda}_j$, and $\mathring{\xi}_j$ as in (3.30) with x replaced by $\mathring{x}$.

Lemma 3.10. *Assume (A1–A2). Let $\alpha \in (\lambda^{-1}, 1)$. Then for $\mathring{g}_0 > 0$ sufficiently small (depending on α), the following hold.*

(i) Uniformly in all $l \le j$,

$$A_j \cdots A_l = \begin{pmatrix} 0 & 0 \\ 0 & O(\mathring{g}_{j+1}^2/\mathring{g}_l^2) \end{pmatrix}. \tag{3.38}$$

(ii) Uniformly in all j,

$$B_j = \begin{pmatrix} 0 & O(\mathring{g}_j\chi_j) \\ 0 & O(\chi_j) \end{pmatrix}. \tag{3.39}$$

(iii) Uniformly in all $l \ge j$,

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} O(1) & 0 \\ O(\chi_j) & O(\alpha^{l-j+1}) \end{pmatrix}. \tag{3.40}$$

Proof. (i) It follows immediately from (3.37) that

$$A_j \cdots A_l = \prod_{k=l}^{j}(1-2\beta_k\mathring{g}_k)\pi_g, \tag{3.41}$$

and thus (2.5) implies (i).

(ii) It follows directly from (3.37) and Lemma 2.2 that (3.39) holds.
(iii) Note that

$$\begin{pmatrix} c_1 & 0 \\ b_1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} c_n & 0 \\ b_n & a_n \end{pmatrix} = \begin{pmatrix} c^* & 0 \\ b^* & a^* \end{pmatrix} \tag{3.42}$$

with

$$a^* = a_1 \cdots a_n, \quad b^* = \sum_{i=1}^{n} a_1 \cdots a_{i-1} b_i c_{i+1} \cdots c_n, \quad c^* = c_1 \cdots c_n. \tag{3.43}$$

We apply this formula with the inverse matrices

$$C_j^{-1} = \begin{pmatrix} (1 - 2\zeta_j \mathring{g}_j)^{-1} & 0 \\ (1 - 2\zeta_j \mathring{g}_j)^{-1} \mathring{\gamma}_j \mathring{\alpha}_j & \mathring{\alpha}_j \end{pmatrix} \tag{3.44}$$

where $\mathring{\alpha}_j = \mathring{\lambda}_j^{-1}$. Thus

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} \mathring{\tau}_{j,l} & 0 \\ \mathring{\sigma}_{j,l} & \mathring{\alpha}_{j,l} \end{pmatrix} \tag{3.45}$$

with

$$\mathring{\alpha}_{j,l} = \mathring{\alpha}_j \cdots \mathring{\alpha}_l, \qquad \mathring{\tau}_{j,l} = \prod_{k=j}^{l} (1 - 2\zeta_k \mathring{g}_k)^{-1}, \tag{3.46}$$

$$\mathring{\sigma}_{j,l} = \sum_{i=1}^{l-j+1} \Big(\prod_{k=j+i}^{l} (1 - 2\zeta_k \mathring{g}_k)^{-1} \Big) \mathring{\gamma}_{j+i-1} \Big(\prod_{k=j}^{j+i-2} \mathring{\alpha}_k \Big). \tag{3.47}$$

The product defining $\mathring{\tau}_{j,l}$ is $O(1)$ by (2.6). Assume that $\mathring{g}_0$ is sufficiently small that, with Lemma 2.2 and (A2), $\mathring{\alpha}_m < \alpha$ for all m. Then $\mathring{\alpha}_{j,l} \le O(\alpha^{l-j+1})$. Similarly, since $\mathring{\gamma}_m \le O(\chi_m)$,

$$|\mathring{\sigma}_{j,l}| \le \sum_{i=1}^{l-j+1} \alpha^i O(\chi_{j+i-1}) \le O(\chi_j). \tag{3.48}$$

This completes the proof. ☐

Proof of Lemma 3.9. The u-component of (3.32) is given by

$$u_{j+1} = A_j u_j + \pi_u r_j. \tag{3.49}$$

By induction, under the initial condition $u_0 = 0$ this recursion is equivalent to

$$u_j = \pi_u y_j = \sum_{l=0}^{j-1} A_{j-1} \cdots A_{l+1} \pi_u r_l, \tag{3.50}$$

which is the same as the u-component of (3.36).

The v-component of (3.32) states that

$$v_{j+1} = B_j u_j + C_j v_j + \pi_v r_j, \tag{3.51}$$

and this is equivalent to

$$v_j = C_j^{-1} v_{j+1} - C_j^{-1} B_j u_j - C_j^{-1} \pi_v r_j. \tag{3.52}$$

By induction, for any $k \geq j$, the latter is equivalent to

$$v_j = C_j^{-1} \cdots C_k^{-1} v_{k+1} - \sum_{l=j}^{k} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l). \tag{3.53}$$

By Lemma 3.10(iii), with some $\alpha \in (\lambda^{-1}, 1)$ and with $\mathring{g}_0$ sufficiently small, $\|C_0^{-1} \cdots C_k^{-1}\|$ is uniformly bounded. Thus, if $y_j = (u_j, v_j)$ satisfies (3.32) and $\|v_j\|_{F_j} \to 0$, then $\|C_0^{-1} \cdots C_k^{-1} v_{k+1}\|_{F_0} \to 0$ and hence

$$v_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l), \tag{3.54}$$

which is the same as the v-component of (3.36). Conversely, suppose that y_j satisfies (3.36) and $\|v_j\|_{F_j} \to 0$. It is also straightforward to conclude that (3.54) implies (3.53) and thus that the v-component of y satisfies (3.32). □

3.4.2 Step 2. Solution of the approximate equation

We now prove existence, uniqueness, and bounds for the solution to the approximate equation (3.32).

Lemma 3.11. *Assume (A1–A2), that* $\mathsf{h}_z, \mathsf{h}_\mu \gg \mathsf{h}_g$, *and* $\mathring{g}_0 \ll 1$. *For each* $r \in X^{\mathsf{r}}$, *there exists a unique solution* $y = Sr \in X^{\mathsf{w}}$ *to* (3.32) *which obeys the boundary conditions* $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$. *Uniformly in small* $\mathring{g}_0$,

$$\|\pi_\alpha S r\|_{X^{\mathsf{w}}} \leq \|r\|_{X^{\mathsf{r}}} \times \begin{cases} 1 & \alpha = K \\ O(1) & \alpha = g, z, \mu. \end{cases} \tag{3.55}$$

The solution map obeys $\pi_K S = S \pi_K$ *and* $\pi_V S = S \pi_V$. *If (A4) holds then all the analogous statements hold in the B-spaces, upon replacing all* X^w *by* BX^w.

Proof. According to Lemma 3.9, it suffices to prove that there is a unique solution in X^{w} to (3.36) (instead of (3.32)) which obeys the boundary conditions. We will prove that $\|\pi_\alpha H\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} \leq O(h_\alpha^{-1} h_g)$ for $\alpha = z, \mu$ (recall that $\pi_\alpha H = 0$ for $\alpha = K, g$), and that $\|\pi_\alpha U r\|_{X^{\mathsf{w}}}$ is bounded by the right-hand side of (3.55). By our assumptions on the h_α, this implies that $\|H\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} \ll 1$ and hence that $(1 - H)^{-1}$ exists on X^{w}, and thus that the unique solution in X^{w} of (3.32) is given by the Neumann series

$$y = Sr = (1 - H)^{-1} U r = \sum_{n=0}^{\infty} H^n U r. \tag{3.56}$$

The boundary condition $\pi_v y_\infty = 0$ is a consequence of $y \in X^{\mathsf{w}}$, and the initial condition $\pi_u y_0 = 0$ is implicit in the equation (3.36). The claim that $\pi_K S = S \pi_K$ and $\pi_V S = S \pi_V$

then follows from (3.35). Since $\pi_u Sr = \pi_u Ur$, the cases $\alpha = K, g$ of (3.55) follow from the bounds claimed for U. Also, the claimed bounds on H imply that $\|(1-H)^{-1}\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le 2$, and with the claimed bounds for Ur, this implies (3.55) also for $\alpha = z, \mu$. Thus, to prove (3.55), it suffices to establish the bounds claimed for H and U.

The extension to the B-spaces is straightforward since all sums converge uniformly in $m \in M_{\mathrm{ext}}$, as we will see, and all summands are continuous in m, by assumption.

To complete the proof, we require estimates for $\pi_\alpha U$ for $\alpha \in \{K, g, z, \mu\}$, and on $\pi_\alpha H$ for $\alpha = z, \mu$. Thus there are six estimates in all. Their treatment is similar, and uses Lemma 2.1(ii), which gives that for all $k \ge j \ge 0$ and $m \ge 0$,

$$\sum_{l=j}^{k} \chi_l \mathring{g}_l^n |\log \mathring{g}_l|^m \le C_{n,m} \begin{cases} |\log \mathring{g}_k|^{m+1} & n = 1 \\ \chi_j \mathring{g}_j^{n-1} |\log \mathring{g}_j|^m & n > 1. \end{cases} \tag{3.57}$$

(i) Bound for K-component. By definition, since $\pi_K A_l = 0$, we have $\pi_K U = \pi_K$. Therefore,

$$\|\pi_K Ur\|_{X^{\mathsf{w}}} \le \sup_j \|\pi_K r_j\|_{X_j^{\mathsf{w}}} \le \sup_j \left[\mathsf{w}_{K,j}^{-1} \mathsf{r}_{K,j}\right] \|r\|_{X^{\mathsf{r}}} = \|r\|_{X^{\mathsf{r}}}. \tag{3.58}$$

(ii) Bound for g-component. By Lemma 3.10(i), (3.11), (2.3), and (3.57),

$$\begin{aligned} \|\pi_g Ur\|_{X^{\mathsf{w}}} &\le \sup_j \sum_{l=0}^{j-1} \|\pi_g A_{j-1} \cdots A_{l+1} r_l\|_{X_j^{\mathsf{w}}} \le \sup_j \sum_{l=0}^{j-1} \mathsf{w}_{g,j}^{-1} \mathsf{r}_{g,l} O(\mathring{g}_j/\mathring{g}_l)^2 \|r\|_{X^{\mathsf{r}}} \\ &\le c\|r\|_{X^{\mathsf{r}}} \sup_j |\log \mathring{g}_j|^{-1} \sum_{l=0}^{j-1} \chi_l \mathring{g}_l \le c\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{3.59}$$

(iii) Bound for z-component. By Lemma 3.10(iii), (3.11), and (3.57),

$$\begin{aligned} \|\pi_z Ur\|_{X^{\mathsf{w}}} &\le \sup_j \sum_{l=j}^{\infty} \|\pi_z C_j^{-1} \cdots C_l^{-1} r_l\|_{X_l^{\mathsf{w}}} \\ &\le c \sup_j \mathsf{h}_z \mathsf{w}_{z,j}^{-1} \sum_{l=j}^{\infty} \chi_l \mathring{g}_l^3 \|r\|_{X^{\mathsf{r}}} \le c |\log \mathring{g}_0|^{-1} \|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{3.60}$$

Similarly, by Lemma 3.10(ii-iii), (3.11), and (3.57),

$$\begin{aligned} \|\pi_z H\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} &\le \sup_j \sum_{l=j}^{\infty} \|\pi_z C_j^{-1} \cdots C_l^{-1} B_l\|_{L(X_l^{\mathsf{w}},X_j^{\mathsf{w}})} \\ &\le c \sup_j \mathsf{w}_{z,j}^{-1} \sum_{l=j}^{\infty} \chi_l \mathring{g}_l \mathsf{w}_{g,l} \le c \mathsf{h}_z^{-1} \mathsf{h}_g. \end{aligned} \tag{3.61}$$

(iv) Bound for μ-component. Using Lemma 3.10(iii), we obtain

$$\begin{aligned}\|\pi_\mu U r\|_{X^{\mathsf{w}}} &= \sup_j \Big[\sum_{l=j}^{\infty} \|\pi_\mu C_j^{-1} \cdots C_l^{-1} r_l\|_{X_j^{\mathsf{w}}} \Big] \\ &\le c \sup_j \mathsf{w}_{\mu,j}^{-1} \Big[\mathsf{h}_z \sum_{l=j}^{\infty} \chi_l \mathring{g}_l^3 + \mathsf{h}_\mu \sum_{l=j}^{\infty} \alpha^{l-j+1} \chi_l \mathring{g}_l^3 \Big] \|r\|_{X^{\mathsf{r}}} \\ &\le c |\log \mathring{g}_0|^{-1} \|r\|_{X^{\mathsf{r}}}, \end{aligned} \tag{3.62}$$

where we used (3.57) and also that $\sum_{l=j}^{\infty} \alpha^{l+1-j} \chi_l \mathring{g}_l^3 \le c \chi_j \mathring{g}_j^3$ in the last step. To bound $\|\pi_\mu H\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})}$, we argue similarly as we did for $\pi_\mu U r$, and use Lemma 3.10 to obtain

$$\begin{aligned}\|\pi_\mu H\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} &\le \sup_j \sum_{l=j}^{\infty} \|\pi_\mu C_j^{-1} \cdots C_l^{-1} B_l\|_{L(X_l^{\mathsf{w}}, X_j^{\mathsf{w}})} \\ &\le c \sup_j \mathsf{w}_{\mu,j}^{-1} \left[\sum_{l=j}^{\infty} \mathring{g}_j \chi_j \mathsf{w}_{g,l} + \sum_{l=j}^{\infty} \alpha^{l+1-j} \chi_j \mathsf{w}_{g,l} \right] \le c \mathsf{h}_\mu^{-1} \mathsf{h}_g. \end{aligned} \tag{3.63}$$

This proves the required bounds for $\alpha = \mu$ and thus completes the proof. ∎

3.4.3 Step 3. Solution of the linear equation

We now prove Lemma 3.6, which involves solving the equation (3.16).

Proof of Lemma 3.6. Fix $\omega \in (\kappa\Omega, 1)$ and $\varepsilon > 0$.
(i) We define

$$\begin{aligned} W_j(t, x_j) &= D_x \Phi_j(t, x_j) - L_j \\ &= [D_x \bar{\varphi}_j^0(x_j) - D_x \bar{\varphi}_j^0(\mathring{x})] + D_x(\psi(x_j), t\rho(x_j)), \end{aligned} \tag{3.64}$$

and rewrite (3.16) as

$$y_{j+1} = D_x \Phi_j(t, x_j) y_j + r_j = L_j y_j + W_j(t, x_j) y_j + r_j. \tag{3.65}$$

We claim that $W : [0,1] \times (\mathring{x} + \mathsf{B}) \to L(X^{\mathsf{w}}, X^{\mathsf{r}})$, that W is continuously Fréchet differentiable, and that if $x \in \mathring{x} + \mathsf{B}$ then

$$\begin{aligned} \|\pi_K W_j(t, x_j)\|_{L(X_j^{\mathsf{w}}, X_{j+1}^{\mathsf{r}})} \le \omega, \quad \|\pi_V W_j(t, x_j)\|_{L(X_j^{\mathsf{w}}, X_{j+1}^{\mathsf{r}})} \le \varepsilon, \\ \|D_x W_j(t, x_j)\|_{L(X_j^{\mathsf{w}}, L(X_j^{\mathsf{w}}, X_{j+1}^{\mathsf{r}}))} \le \varepsilon, \end{aligned} \tag{3.66}$$

and that the analogous statements hold in the B-spaces. To see this, note that the first term on the right-hand side of (3.64) only depends on the V-components, and is continuously Fréchet differentiable since, by (3.29), $D^2 \bar{\Phi}_j$ is constant for each j with coefficients bounded by $O(\chi_j)$. Therefore, for $x \in \mathring{x} + \mathsf{B}$, with the notation $\mathsf{r}_{V,j+1}^{-1} = \max_{\alpha=g,z,\mu} \mathsf{r}_{\alpha,j+1}^{-1}$,

$$\begin{aligned} \|[D\bar{\varphi}_j^0(\mathring{x}_j) - D\bar{\varphi}_j^0(x_j)]\pi_V\|_{L(X_j^{\mathsf{w}}, X_{j+1}^{\mathsf{r}})} &\le c\chi_j \mathsf{r}_{V,j+1}^{-1} \mathsf{w}_{V,j}^2 \|\mathring{x}_j - x_j\|_{X_j^{\mathsf{w}}} \\ &\le c\chi_j \mathsf{r}_{V,j+1}^{-1} \mathsf{w}_{V,j}^2 = O(\mathring{g}_0 |\log \mathring{g}_0|^2). \end{aligned} \tag{3.67}$$

This contributes to the second bound in (3.66), with $\mathring{g}$ taken small. The second term on the right-hand side of (3.64), as well as its derivative, have been bounded in Lemma 3.7 (where we take ε sufficiently small). This completes the proof of (3.66).

By the assumption that $y \in X^{\mathsf{w}}$, Lemma 3.11, and (3.66), the equation (3.65) with the boundary conditions of Lemma 3.6(i) is equivalent to

$$y = S(W(t,x)y + r). \tag{3.68}$$

We apply Lemma 3.11 (in particular $\pi_K S = S\pi_K$, $\pi_V S = S\pi_V$), and (3.66) with ε sufficiently small to obtain

$$\begin{aligned} &\|\pi_K SW(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le \|\pi_K S\|_{L(X^{\mathsf{r}},X^{\mathsf{w}})}\|\pi_K W(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le \omega, \\ &\|\pi_V SW(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le \|\pi_V S\|_{L(X^{\mathsf{r}},X^{\mathsf{w}})}\|\pi_V W(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le O(1)\varepsilon \le \omega. \end{aligned} \tag{3.69}$$

Thus, since $\omega < 1$, the solution operator $S(t,x)$ is given in terms of S and W by the Neumann series

$$S(t;x) = (1 - SW(t,x))^{-1}S = \sum_{n=0}^{\infty}(SW(t,x))^n S. \tag{3.70}$$

(ii) To bound the norm of the solution operator, we apply (3.70), (3.69), and Lemma 3.11 to obtain

$$\|S(t,x)r\|_{X^{\mathsf{w}}} \le (1-\omega)^{-1}\|Sr\|_{X^{\mathsf{w}}} \le C\|r\|_{X^{\mathsf{r}}}. \tag{3.71}$$

(iii) It is shown in (3.70) that $S(t,x) = (1-SW(t,x))^{-1}S$. Thus, Fréchet differentiability of $S(t,x)$ follows from the Fréchet differentiability of $SW(t,x)$, which itself follows from part (i) and the fact that $D_x SW(t,x) = SD_x W(t,x)$ due to the linearity of S. Explicitly,

$$D_x S(t,x) = (1 - SW(t,x))^{-1} D_x SW(t,x)(1 - SW(t,x))^{-1}S. \tag{3.72}$$

By (3.66),

$$\|D_x SW(t,x)\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{w}}))} \le C\|D_x W(t,x)]\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{r}}))} \le \varepsilon'. \tag{3.73}$$

By combining this with (3.69), and by choosing ε' sufficiently small, we then obtain

$$\begin{aligned} \|D_x S(t,x)r\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} &\le (1-\omega)^{-2}\varepsilon'\|Sr\|_{X^{\mathsf{w}}} \\ &\le C(1-\omega)^{-2}\varepsilon'\|r\|_{X^{\mathsf{r}}} \le \varepsilon\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{3.74}$$

This proves (3.18) and completes the proof. □

3.5 Proof of Lemmas 3.4 and 3.8

Proof of Lemma 3.4. Suppose first that (A3') holds. Then

$$\|\psi_j(x_j)\|_{\mathcal{W}_{j+1}} \le \|\psi_j(\tilde{x}_j)\|_{\mathcal{W}_{j+1}} + \sup_{s\in[0,1]} \|D\psi_j(\tilde{x}_j + s(x_j - \tilde{x}_j))(x_j - \tilde{x}_j)\|_{\mathcal{W}_{j+1}}. \tag{3.75}$$

Since $\|K_j - \tilde{K}_j\|_{\mathcal{W}_j} = O(\chi_j \mathring{g}_j^3)$ and $\|V_j - \tilde{V}_j\|_{\mathcal{V}} = O(\mathring{g}_j^2 |\log \mathring{g}_j|) = O(\mathring{g}_j)$ for $x \in \mathring{x} + \mathsf{B}$, with (1.23) we obtain

$$\|\psi_j(x_j)\|_{\mathcal{W}_{j+1}} \le O(\chi_j \mathring{g}_j^3) + \kappa \|K_j - \tilde{K}_j\|_{\mathcal{W}_j} + O(\chi_j \mathring{g}_j^2)\|V_j - \tilde{V}_j\|_{\mathcal{V}} \le O(\chi_j \mathring{g}_j^3) \tag{3.76}$$

as claimed.

Now assume (A3). Since $\psi_j(0) = 0$,

$$\|\psi_j(x_j)\|_{\mathcal{W}_{j+1}} \le \sup_{s \in [0,1]} \|D\psi_j(s x_j) x_j\|_{\mathcal{W}_{j+1}}. \tag{3.77}$$

Hence, by (A3) and Lemma 2.2, there is a $C > 0$ such that

$$\|\mathring{K}_{j+1}\|_{\mathcal{W}_{j+1}} \le \kappa \|\mathring{K}_j\|_{\mathcal{W}_j} + O(\chi_j \mathring{g}_j^2)\|\bar{V}_j\| \le \kappa \|\mathring{K}_j\|_{\mathcal{W}_j} + C \chi_j \mathring{g}_j^3. \tag{3.78}$$

We make the inductive assumption $\|\mathring{K}_j\|_{\mathcal{W}_j} \le M \chi_j \mathring{g}_j^3$, which holds for $j = 0$ as long as M is large enough, by hypothesis. This leads to

$$\|\mathring{K}_{j+1}\|_{\mathcal{W}_{j+1}} \le (\kappa M + C)\chi_j \mathring{g}_j^3 \le (\kappa M + C)\Omega(1 + O(\mathring{g}_0))\chi_{j+1} \mathring{g}_{j+1}^3. \tag{3.79}$$

The right-hand side is at most $M \chi_{j+1} \mathring{g}_{j+1}^3$, provided we choose $\mathring{g}_0$ small enough that $\kappa\Omega(1 + O(\mathring{g}_0)) < 1$ (which is possible because $\kappa\Omega < 1$ by (A3) or (A3')), and choose M sufficiently large. This completes the proof. □

Proof of Lemma 3.8. Let $\mathsf{B}_\theta = (1 - \theta)\mathsf{B}$ and

$$\bar{\mathsf{I}} = ([\tfrac{1}{2}\mathring{g}_0, 2\mathring{g}_0] \times \mathcal{W}_0) \cap \bar{x}^{-1}(\mathring{x} + \mathsf{B}_\theta). \tag{3.80}$$

We will show that $\bar{\mathsf{I}}$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ and that $\bar{x} : \bar{\mathsf{I}} \to \mathring{x} + \mathsf{B}_\theta$ is continuously Fréchet differentiable. Since $\bar{x}^{-1}(\mathring{x} + \mathsf{B}_\theta) = \bar{V}^{-1}(\mathring{x} + \mathsf{B}_\theta) \cap \bar{K}^{-1}(\mathring{x} + \mathsf{B}_\theta)$, it suffices to show that each of $\bar{V}^{-1}(\mathring{x} + \mathsf{B}_\theta)$ and $\bar{K}^{-1}(\mathring{x} + \mathsf{B}_\theta)$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$, and that each of $\bar{V}$ and $\bar{K}$ is continuously Fréchet differentiable on $\bar{\mathsf{I}}$.

We begin with $\bar{V}$. Let $\bar{V}_j'$ denote the derivative of $\bar{V}_j$ with respect to g_0, and let $\bar{V}' = (\bar{V}_j')$ denote the sequence of derivatives. It is straightforward to conclude from Lemmas 2.3 and 2.1(iv) that $\bar{V}' \in X^{\mathsf{w}}$ if $g_0 \in \bar{\mathsf{I}}_g \subseteq [\frac{1}{2}\mathring{g}_0, 2\mathring{g}_0]$ and similarly that $\bar{V}^{-1}(\mathring{x} + \mathsf{B}_\theta)$ contains a neighbourhood of $\mathring{g}$. That $\bar{V}'$ is, in fact, the Fréchet derivative of $\bar{V}$ in the space X^{w} can be deduced from the fact that the sequence $\bar{V}''(g_0)$ is uniformly bounded in X^{w} for $g_0 \in \bar{\mathsf{I}}_g$ (though not uniform in $\mathring{g}_0$). In fact, by Lemma 2.3,

$$\|\bar{V}_j(g_0 + \varepsilon) - \bar{V}_j(g_0) - \varepsilon \bar{V}_j'(g_0)\|_{X_j^{\mathsf{w}}} \le O(\varepsilon^2) \sup_{0 < \varepsilon' < \varepsilon} \|\bar{V}_j''(g + \varepsilon')\|_{X_j^{\mathsf{w}}}. \tag{3.81}$$

The continuity of $\bar{V}'$ in X^{w} follows similarly.

For $\bar{K}$, we first note that $\|D_{K_0}\bar{K}_0\|_{L(\mathcal{W}_0, \mathcal{W}_0)} = 1$, $\|D_{g_0}\bar{K}_0\|_{\mathcal{W}_0} = 0$. By (A3) or (A3') and by induction,

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathcal{W}_0, \mathcal{W}_{j+1})} \le \kappa \|D_{K_0}\bar{K}_j\|_{L(\mathcal{W}_0, \mathcal{W}_j)} \le \kappa^{j+1}. \tag{3.82}$$

Since $\kappa < \Omega^{-1} < 1$, and since $\mathring{g}_{j+1}/\mathring{g}_j \to 1$ by (2.3), we obtain

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathcal{W}_0,\mathcal{W}_{j+1})} \leq O(\mathsf{w}_{K,j+1}), \tag{3.83}$$

where the constant may depend on $\mathring{g}_0$. Similarly, by (1.11) and Lemma 2.3,

$$\begin{aligned}\|D_{g_0}\bar{K}_{j+1}\|_{\mathcal{W}_{j+1}} &\leq \kappa\|D_{g_0}\bar{K}_j\|_{\mathcal{W}_j} + O(\chi_j\bar{g}_j^2)\|D_{g_0}\bar{V}_j\|_{\mathcal{V}} \\ &\leq \kappa\|D_{g_0}\bar{K}_j\|_{\mathcal{W}_j} + O(\chi_j\bar{g}_j^4/\bar{g}_0^2).\end{aligned} \tag{3.84}$$

By induction as in the proof of Lemma 3.4, again using $\kappa < \Omega^{-1}$, we conclude

$$\|D_{g_0}\bar{K}_{j+1}\|_{\mathcal{W}_{j+1}} \leq O(\chi_j\bar{g}_j^4/\bar{g}_0^2) \leq O(\mathsf{w}_{K,j+1}). \tag{3.85}$$

These bounds imply that $\bar{K}^{-1}(\mathring{x} + \mathsf{B}_\theta)$ contains a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ and also that the component-wise derivatives of $\bar{K}$ with respect to g_0 and K_0 are respectively in $X^{\mathsf{w}} \cong L(\mathbb{R}, X^{\mathsf{w}})$ and $L(\mathcal{W}_0, X^{\mathsf{w}})$.

To verify that the component-wise derivative is the Fréchet derivative in X^{w} of $\bar{K}$, it again suffices to obtain bounds on the second derivatives in X^{w}, as in (3.81). For example, since $D_{K_0}^2\bar{K}_0 = 0$, $D_{K_0}\bar{V}_j = 0$, and

$$D_{K_0}^2\bar{K}_{j+1} = D_K\psi(\bar{K}_j,\bar{V}_j)D_{K_0}^2\bar{K}_j + D_K^2\psi(\bar{K}_j,\bar{V}_j)D_{K_0}\bar{K}_jD_{K_0}\bar{K}_j, \tag{3.86}$$

it follows from (3.82) and induction that, for $(K_0, g_0) \in \bar{\mathsf{I}}$ with $\bar{\mathsf{I}} \subset \mathcal{W}_0 \oplus \mathbb{R}$ chosen sufficiently small,

$$\|D_{K_0}^2\bar{K}_{j+1}\| \leq \kappa\|D_{K_0}^2\bar{K}_j\| + C\kappa^{2j} \leq C(1 + j\kappa)\kappa^j \leq O(\mathsf{w}_{K,j+1}) \tag{3.87}$$

and thus that the component-wise derivative $D_{K_0}^2\bar{K}$ is uniformly bounded in $L^2(\mathcal{W}_0, X^{\mathsf{w}})$ for $(K_0, g_0) \in \bar{\mathsf{I}}$. Similarly, slightly more complicated recursion relations than (3.86) for $D_{g_0}^2\bar{K}_j$ and $D_{g_0}D_{K_0}\bar{K}_j$ show that the component-wise second derivative of $\bar{K}$ is uniformly bounded in $L^2(\mathcal{W}_0 \oplus \mathbb{R}, X^{\mathsf{w}})$ for $\bar{\mathsf{I}}$ sufficiently small. This shows as in (3.81) that $\bar{K}$, and thus $\bar{x}$, is continuously Fréchet differentiable from $\bar{\mathsf{I}}$ to X^{w}.

If (A4) holds, all of the above estimates are uniform in the external parameter, and it can be seen from this that the claim holds with X^w replaced by BX^w. This completes the proof. $\square$

3.6 Proof of Lemma 3.7

We use the notation

$$\begin{aligned}\mathsf{w}_{V,j} &= \max\{\mathsf{w}_{g,j}, \mathsf{w}_{z,j}, \mathsf{w}_{\mu,j}\}, \\ \mathsf{r}_{V,j}^{-1} &= \max\{\mathsf{r}_{g,j}^{-1}, \mathsf{r}_{z,j}^{-1}, \mathsf{r}_{\mu,j}^{-1}\}, \quad \mathsf{r}_j^{-1} = \max\{\mathsf{r}_{V,j}^{-1}, \mathsf{r}_{K,j}^{-1}\}.\end{aligned} \tag{3.88}$$

Proof of Lemma 3.7. We begin with the bounds on the first derivatives in (3.20)–(3.21). By assumptions (1.10)–(1.11), together with (2.3), the definition of the weights (3.11),

and for (3.90) also the fact that $\chi_j/\chi_{j+1} \le \Omega$ by (1.9), we obtain

$$\|D\psi_j(x_j)\pi_V\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le C\chi_j \mathring{g}_j^2 \mathsf{r}_{K,j+1}^{-1}\mathsf{w}_{V,j} \le O(\mathring{g}_0|\log \mathring{g}_0|), \tag{3.89}$$

$$\|D\psi_j(x_j)\pi_K\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le \kappa \mathsf{r}_{K,j+1}^{-1}\mathsf{w}_{K,j} \le \kappa\Omega(1+O(\mathring{g}_0)), \tag{3.90}$$

$$\|D\rho_j(x_j)\pi_V\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le C\chi_j \mathring{g}_j^2 \mathsf{r}_{V,j+1}^{-1}\mathsf{w}_{V,j} \le O(\mathring{g}_0|\log \mathring{g}_0|), \tag{3.91}$$

$$\|D\rho_j(x_j)\pi_K\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le C \mathsf{r}_{V,j+1}^{-1}\mathsf{w}_{K,j} \le O(\mathsf{h}_V^{-1}\mathsf{h}_K). \tag{3.92}$$

This establishes the bounds on the first derivatives in (3.20)–(3.21).

Let ϕ denote either ψ or ρ. Then similarly, for $m \in \{2,3\}$, (1.12)–(1.13) imply $\|D^m\phi_j\|_{L^m(X_j,X_{j+1})} \le O(1)$ and $\|D_V^m\phi_j\|_{L^m(X_j,X_{j+1})} \le O(\chi_j)$. By (3.11), $\chi_j \mathsf{r}_{j+1}^{-1}\mathsf{w}_{V,j}^m$ and $\mathsf{r}_{j+1}^{-1}\mathsf{w}_K \mathsf{w}_j^{m-1}$ are both $O(\mathring{g}_0|\log \mathring{g}_0|^2)$, which in particular gives the bounds on the second derivatives in (3.20)–(3.21). In addition, these bounds on the second and third derivatives imply that

$$\|\phi(x+y) - \phi(x) - D\phi(x)y\|_{X^{\mathsf{r}}} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.93}$$

$$\|D\phi(x+y) - D\phi(x) - D^2\phi(x)y\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.94}$$

and hence that $\phi : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ is twice Fréchet differentiable. The above bound on the third derivatives implies continuity of this differentiability.

It remains to verify the bound on $\rho(x)$ in (3.20). For $x = (x_j) \in \mathring{x} + \mathsf{B}$, we write $x_j = (K_j, V_j)$. We first assume (A3). By Lemmas 2.2 and 3.4,

$$\|K_j\|_{\mathcal{W}_j} \le \|\mathring{K}_j\|_{\mathcal{W}_j} + \mathsf{h}_K \chi_j \mathring{g}_j^3 \le (C + \mathsf{h}_K)\chi_j \mathring{g}_j^3 \le O(\mathsf{h}_K \chi_j \mathring{g}_j^3), \tag{3.95}$$

$$\|V_j\|_{\mathcal{V}} \le \|\mathring{V}_j\|_{\mathcal{V}} + \mathsf{h}_V \mathring{g}_j^2 |\log \mathring{g}_j| \le O(\mathring{g}_j), \tag{3.96}$$

with the constant in (3.96) taken independent of h_V by choosing $\mathring{g}_0$ sufficiently small (depending on h_V). The assumption $\rho_j(0) = 0$ implies

$$\|\rho_j(x_j)\|_{\mathcal{V}} \le \sup_{s\in[0,1]} \|D\rho_j(sx_j)x_j\|_{\mathcal{V}}. \tag{3.97}$$

With (1.10)–(1.13), the above estimates imply that

$$\|\rho_j(x_j)\|_{X_{j+1}^{\mathsf{r}}} \le M \mathsf{r}_{V,j}^{-1}\|K_j\|_{\mathcal{W}_j} + M\chi_j \|V_j\|_{\mathcal{V}}^2 \mathsf{r}_{V,j}^{-1}\|V_j\|_{\mathcal{V}} \le O(\mathsf{h}_K \mathsf{h}_V^{-1}). \tag{3.98}$$

which is less than ε for $\mathsf{h}_g, \mathsf{h}_z, \mathsf{h}_\mu$ sufficiently large.

Finally, under (A3'),

$$\|\rho_j(x_j)\|_{X_{j+1}^{\mathsf{r}}} \le \|\rho(\tilde{x}_j)\|_{X_{j+1}^{\mathsf{r}}} + \sup_{s\in[0,1]} \|D\rho_j(\tilde{x}_j + s(x_j - \tilde{x}_j))(x_j - \tilde{x}_j)\|_{X_{j+1}^{\mathsf{r}}}, \tag{3.99}$$

which is less than ε for $\mathsf{h}_g, \mathsf{h}_z, \mathsf{h}_\mu$ sufficiently large. This completes the proof. ☐

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